Sachs-Wolfe Effect - Gravitational Potential ϕ, Integrated Sachs-Wolfe

The Gravitational Potential $\Phi(a)$

1. Definition and Gauge Choice

In linear cosmological perturbation theory, scalar metric perturbations about a Friedmann–Robertson–Walker background may be written in Newtonian (longitudinal) gauge as

$$ds^2 = -(1 + 2\Phi)\, dt^2 + a^2(t)(1 - 2\Psi)\, \delta_{ij}\, dx^i dx^j .$$

Here $\Phi(\mathbf{x}, t)$ and $\Psi(\mathbf{x}, t)$ represent the gravitational potentials associated with **time dilation and spatial curvature**, respectively. In the absence of significant anisotropic stress—as is an excellent approximation after recombination in ΛCDM—these two potentials are equal,

$$\Phi = \Psi.$$

The potential Φ plays the role of the **relativistic generalization of the Newtonian gravitational potential**, governing both the motion of non-relativistic matter and the gravitational redshift experienced by photons.

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2. Physical Meaning of Φ

The gravitational potential Φ measures the depth of spacetime curvature wells created by density perturbations. Its physical effects include:

- **Time dilation:** clocks in potential wells run more slowly.
- **Gravitational redshift:** photons climbing out of wells lose energy.
- **Structure formation:** gradients of Φ accelerate matter, driving growth of overdensities.

For the Cosmic Microwave Background, Φ determines how much energy CMB photons lose or gain both at last scattering and during their journey to the observer.

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3. Relation to Density Perturbations

On sub-horizon scales, **Φ is related to the matter density contrast δ through a relativistic Poisson equation,**

$$\nabla^2 \Phi = 4\pi G a^2 \bar{\rho}\, \delta,$$

which shows that Φ encodes the gravitational influence of matter perturbations. On super-horizon scales, Φ remains well defined and nearly constant for adiabatic initial conditions.

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4. Evolution of Φ Across Cosmological Epochs

Radiation Domination ($a \ll a_{\text{eq}}$)

During radiation domination, relativistic pressure resists gravitational collapse. As perturbations enter the horizon, gravitational potentials decay. This early-time evolution is responsible for the **early Integrated Sachs–Wolfe effect**, though it contributes only modestly to the observed CMB anisotropy.

Matter Domination ($a_{\text{eq}} \lesssim a \ll a_\Lambda$)

Once matter dominates the energy density of the universe, density perturbations grow as

$$D(a) \propto a,$$

and the gravitational potential becomes approximately constant:

$$\Phi(a) \propto \frac{D(a)}{a} \approx \text{constant}.$$

This constancy is a key result: it implies that photons traveling through matter-dominated spacetime experience **no net gravitational redshift along the line of sight**, and hence the Integrated Sachs–Wolfe effect vanishes during this epoch

Dark-Energy Domination ($a \gtrsim a_\Lambda$)
When dark energy begins to dominate, the expansion of the universe accelerates. The growth of structure slows, and density perturbations no longer grow as fast as a. Consequently,

$$\Phi(a) \propto \frac{D(a)}{a} \text{ decays with time.}$$

This decay of gravitational potentials is the physical origin of the **late-time Integrated Sachs–Wolfe effect**. Photons traversing decaying potential wells gain a small net amount of energy, producing additional temperature anisotropy on the largest angular scales.

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5. Normalization and Interpretation of the Plot
In the figure, the potential is plotted in normalized form:

$$\Phi_{\text{norm}}(a) = \frac{\Phi(a)}{\Phi(a=1)}.$$

This normalization emphasizes **relative evolution** rather than absolute amplitude:

- $\Phi_{\text{norm}}(1) = 1$by construction.
- Deviations from unity at earlier times indicate how much deeper gravitational wells were in the past.
- The slope $d\Phi/da$directly controls the magnitude of the ISW contribution.

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6. Connection to the Sachs–Wolfe and ISW Effects
The gravitational potential enters the observed CMB temperature anisotropy through

$$\frac{\Delta T}{T} = \frac{1}{3}\Phi(\eta_{\text{ls}}) + 2\int_{\eta_{\text{ls}}}^{\eta_0} \dot{\Phi}(\eta)\, d\eta.$$

- The **ordinary Sachs–Wolfe term** depends on the value of Φat last scattering.
- The **Integrated Sachs–Wolfe term** depends on the time derivative $\dot{\Phi}$.

Thus, the potential evolution shown in the figure is not merely a background quantity—it is the **direct source** of large-scale CMB anisotropies.

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7. Why the Gravitational Potential Is a Sensitive Probe of Dark Energy
Unlike the expansion rate alone, the gravitational potential responds to both the background cosmology and the growth of structure. Its late-time decay provides:

- A test of cosmic acceleration independent of supernovae,
- A link between the CMB and large-scale structure via ISW cross-correlations,
- Sensitivity to deviations from ΛCDM, such as evolving dark energy or modified gravity.

For this reason, $\Phi(a)$occupies a central role in modern observational cosmology.

8. Summary
The gravitational potential $\Phi(a)$encodes the depth and evolution of spacetime curvature generated by density perturbations. Its near constancy during matter domination suppresses line-of-sight temperature shifts, while its decay during dark-energy domination generates the Integrated Sachs–Wolfe effect. The evolution of $\Phi(a)$therefore provides a direct physical bridge between the expansion history of the universe and the large-scale structure of the Cosmic Microwave Backgroun

Evolution of the Gravitational Potential $\Phi(a)$
Definition
The Newtonian-gauge gravitational potential Φdescribes scalar perturbations of the metric in the linear regime. In the absence of anisotropic stress, it governs both gravitational redshift and the growth of structure.
The quantity plotted is

$$\boxed{\Phi_{\text{norm}}(a) = \frac{\Phi(a)}{\Phi(a=1)},}$$

so that the potential is normalized to unity at the present epoch.

Model for $\Phi(a)$
The potential is related to the linear growth factor $D(a)$by

$$\Phi(a) \propto \frac{D(a)}{a}.$$

The growth factor is approximated by

$$D(a) = \frac{5\Omega_m}{2}\, E(a) \int_0^a \frac{da^{'}}{a^{'3} E^3(a^{'})}, E(a) = \sqrt{\frac{\Omega_r}{a^4} + \frac{\Omega_m}{a^3} + \Omega_\Lambda},$$

and is normalized such that $D(1) = 1$.

Physical Interpretation

- **Matter domination**
When the universe is matter-dominated, $D(a) \propto a$, so $\Phi(a)$remains approximately constant and $\dot{\Phi} \approx 0$. In this case, the Integrated Sachs–Wolfe effect vanishes.
- **Dark-energy domination**
As dark energy becomes dynamically important at late times, the growth of structure slows, causing gravitational potentials to decay. This time dependence ($\dot{\Phi} \neq 0$) generates the ISW contribution seen in the low-multipole power spectrum.

Connection to the Figure Above
The two figures are directly linked:

- **Figure B** shows the **cause**: time evolution of gravitational potentials.
- **Figure A** shows the **effect**: large-scale temperature anisotropies imprinted on the CMB.

A constant $\Phi(a)$produces a flat Sachs–Wolfe plateau, while a decaying $\Phi(a)$leads to enhanced power at the lowest multipoles.

Summary Description Sachs–Wolfe
Here ***Φ(ηls)*** denotes the Newtonian-gauge gravitational potential evaluated at the conformal time of last scattering, which determines the ordinary Sachs–Wolfe temperature anisotropy through $(\Delta T/T)_{SW} = \Phi(\eta_{ls})/3$.

$GP := READPRN(\text{"Phi\_of\_a\_normalized.csv"})$

$a := GP^{\langle 0 \rangle}$ $\qquad$ $\phi_{norm} := GP^{\langle 1 \rangle}$

$$\boxed{\Phi_{\text{norm}}(a) = \frac{\Phi(a)}{\Phi(a=1)}}$$

Φ(a) Newtonian-gauge gravitational potential evaluated at the time of last scattering z~ 1100. **Φ:** scalar metric perturbation describing gravitational potential wells

Time Domain (Cosmic History) Plot - Not Angular

The ISW Effect is controlled by $\dot{\Phi} \neq 0$

So in the $\Phi(a)$ plot:

- **ISW-active regions** → where $\Phi(a)$ $\dot{\Phi} \neq 0$. d (changing)
- **No ISW** → where $\Phi(a)$ is **flat**

Visual rule of thumb

- Flat $\Phi(a)$ → no ISW
- Decaying $\Phi(a)$ → ISW present

In ΛCDM, the gravitational potential is computed via the growth factor:

Growth factor

$$\boxed{\Phi(a) \propto \frac{D(a)}{a}}$$

$$D(a) = \frac{5\Omega_m}{2} E(a) \int_0^a \frac{da'}{a'^3 E^3(a')} \quad \text{with} \quad E(a) = \sqrt{\frac{\Omega_r}{a^4} + \frac{\Omega_m}{a^3} + \Omega_\Lambda}$$

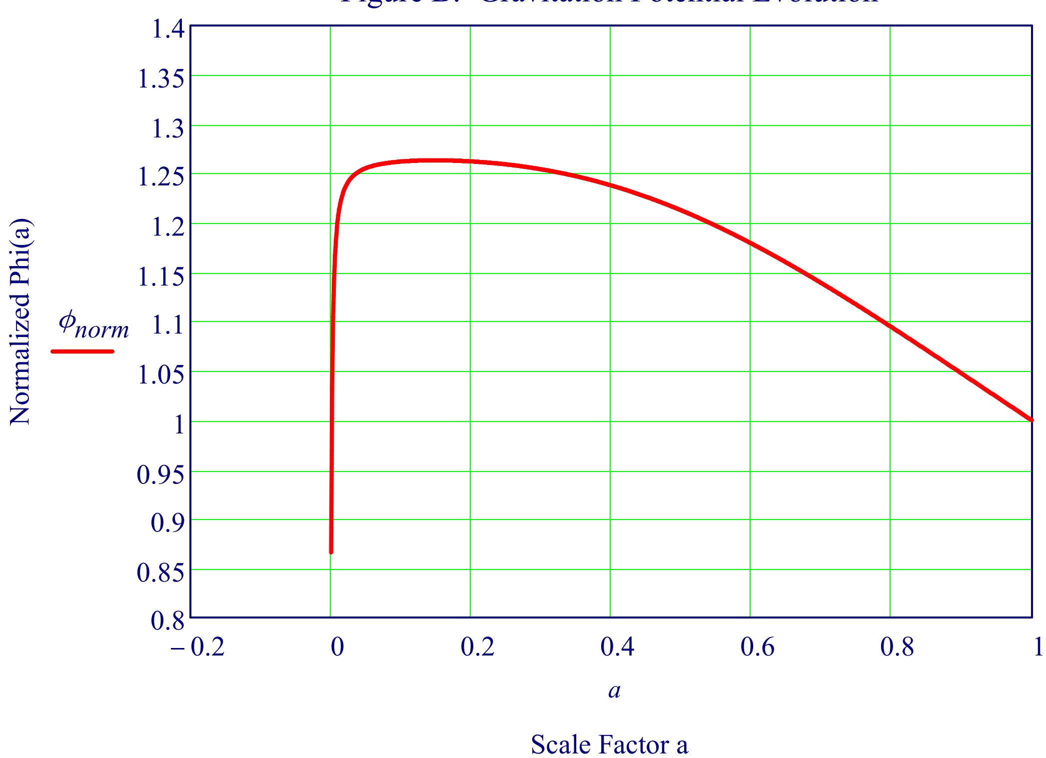
In the line-of-sight formalism, the Sachs–Wolfe contribution to the temperature transfer function is:

$$\boxed{\Delta_\ell^{\text{SW}}(k) = \frac{1}{3}\,\Phi(\eta_{\text{ls}})\, j_\ell(k r_{\text{ls}})}$$

j_l: spherical Bessel function

τ_{ls}: comoving distance to last scattering.

Figure B: Gravitation Potential Evolution



Integrated Sachs - Wolfe (ISW) Effect

Low-Multipole CMB Temperature Power Spectrum (Sachs–Wolfe + ISW)
Definition
The angular power spectrum of CMB temperature anisotropies is defined by the spherical-harmonic expansion

$$\frac{\Delta T}{T}(\hat{\mathbf{n}}) = \sum_{\ell m} a_{\ell m}\, Y_{\ell m}(\hat{\mathbf{n}}),$$

with

$$\langle a_{\ell m} a^{*}_{\ell' m'} \rangle = C_\ell\, \delta_{\ell\ell'}\, \delta_{mm'}.$$

The quantity plotted is

$$\boxed{D_\ell \equiv \frac{\ell(\ell+1)}{2\pi}\, C_\ell,}$$

which represents the contribution to the temperature variance per logarithmic interval in multipole moment.

Physical Content
At large angular scales ($\ell \lesssim 30$), temperature anisotropies are dominated by gravitational redshift effects rather than photon–baryon microphysics. Two mechanisms contribute:

1. **Ordinary Sachs–Wolfe effect**

$$\left(\frac{\Delta T}{T}\right)_{\text{SW}} = \frac{1}{3}\,\Phi(\eta_{\text{ls}}),$$

arising from photons climbing out of gravitational potential wells at the surface of last scattering.

2. **Integrated Sachs–Wolfe (ISW) effect**

$$\left(\frac{\Delta T}{T}\right)_{\text{ISW}} = 2\int_{\eta_{\text{ls}}}^{\eta_0} \dot{\Phi}\,(\eta)\, d\eta,$$

generated when gravitational potentials evolve with time as photons propagate to the observer.

Computed Spectrum
In line-of-sight form, the transfer function entering the power spectrum is

$$\Delta_\ell(k) = \frac{1}{3}\Phi(\eta_{\text{ls}})\, j_\ell \quad (kr_{\text{ls}}) + 2\int_{\eta_{\text{ls}}}^{\eta_0} \dot{\Phi}\,(\eta)\, j_\ell \quad (kr(\eta))\, d\eta,$$

leading to

$$C_\ell = 4\pi \int \frac{dk}{k}\, P_\Phi(k)\, \Delta_\ell^2(k),$$

where $j_\ell$are spherical Bessel functions and $P_\Phi\,(k)$is the primordial potential power spectrum.

<u>The Sachs–Wolfe plateau is:</u> $\boxed{\ell(\ell+1)C_\ell \;\approx\; \text{constant} \quad \text{for } \ell \lesssim 30}$

Normalization
Because the absolute amplitude is not fixed in this reduced model, the spectrum is **normalized to the Sachs–Wolfe plateau**:

$$\boxed{D_\ell^{\text{norm}} = \frac{D_\ell}{\langle D_\ell \rangle_{10 \le \ell \lesssim 30}}.}$$

This normalization sets the mean plateau level to unity and highlights deviations due to the Integrated Sachs–Wolfe effect.

Interpretation

- A **nearly flat plateau** at low ℓ reflects scale-invariant primordial perturbations.
- Any **enhancement at the lowest multipoles** arises from late-time decay of gravitational potentials, providing a direct observational signature of dark energy.
- Acoustic peaks are absent because photon–baryon oscillations are not included in the Sachs–Wolfe treatment.

Plots Generated by Python Program:

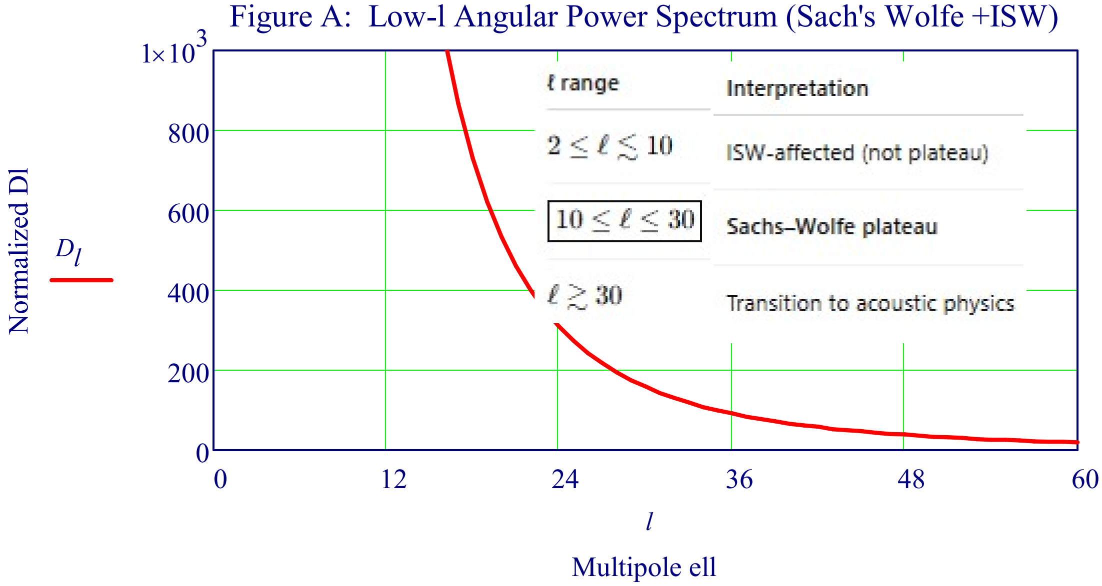
Sachs–Wolfe + ISW - TWO PLOTS - Norm.py
Program Located at: VXPhysics.com/Python

$SW := READPRN$("SW_ISW Normalized.csv")

Read Values: $l := SW^{\langle 0 \rangle}$ $D_l := SW^{\langle 1 \rangle}$

$$D_\ell = \frac{\ell(\ell+1)C_\ell}{2\pi} \quad \text{vs. } \ell$$

Figure A: Low-l Angular Power Spectrum (Sach's Wolfe +ISW)



Summary:

The large-scale CMB temperature anisotropy arises from gravitational redshift at last scattering and from the subsequent evolution of gravitational potentials along the photon path. The Sachs Wolfe plateau reflects nearly scale-invariant primordial perturbations, while any low-multipole enhancement originates from the Integrated Sachs Wolfe effect driven by late-time cosmic acceleration. Together, the angular power spectrum and the gravitational potential history provide a direct link between the CMB and the expansion history of the universe.