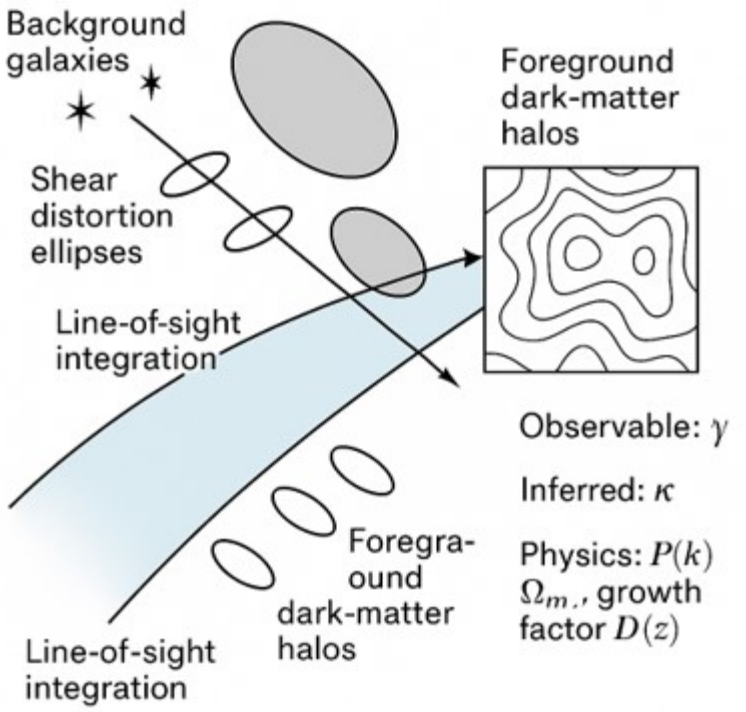
Weak Gravitational Lensing



Core Observable Convergence $\kappa(\theta)$

κ is a weighted line-of-sight projection of the matter overdensity

- Depends on: Ω_m
- Growth of structure
- Geometry via distances
- Scale factor a(χ)

This connects weak lensing directly to ΛCDM parameters, not just observationally but mathematically.

Weak Gravitational Lensing - Key Equations

Convergence (line-of-sight projection):

$$\kappa(\theta) = \frac{3H_0^2\Omega_m}{2c^2}\int_0^{\chi_s} d\chi\, \frac{(\chi_s-\chi)\chi}{\chi_s\, a(\chi)}\, \delta(\chi,\theta)$$

Shear–convergence relation (Fourier space, flat-sky):

$$\gamma(\ell) = e^{2i\varphi_\ell}\, \kappa(\ell)$$

Convergence angular power spectrum (Limber approximation, flat universe):

$$C_\ell^{\kappa\kappa} = \int_0^{\chi_s} d\chi\, \frac{W^2(\chi)}{\chi^2}\, P_\delta\!\left(k = \frac{\ell+1/2}{\chi},\, z(\chi)\right)$$

Lensing efficiency (single source plane):

$$W(\chi) = \frac{3H_0^2\Omega_m}{2c^2}\, \frac{\chi}{a(\chi)}\, \frac{\chi_s-\chi}{\chi_s}$$

Rescaled spectrum often plotted:

$$D_\ell^{\kappa\kappa} \equiv \frac{\ell(\ell+1)}{2\pi}\, C_\ell^{\kappa\kappa}$$

S8 parameter (growth-of-structure diagnostic):

$$S_8 \equiv \sigma_8 \sqrt{\Omega_m/0.3}$$

Key Equations

Generic observable–parameter mapping:

$$\mathcal{O} = \mathcal{F}(\Omega_m, H_0, \sigma_8, w, \ldots)$$

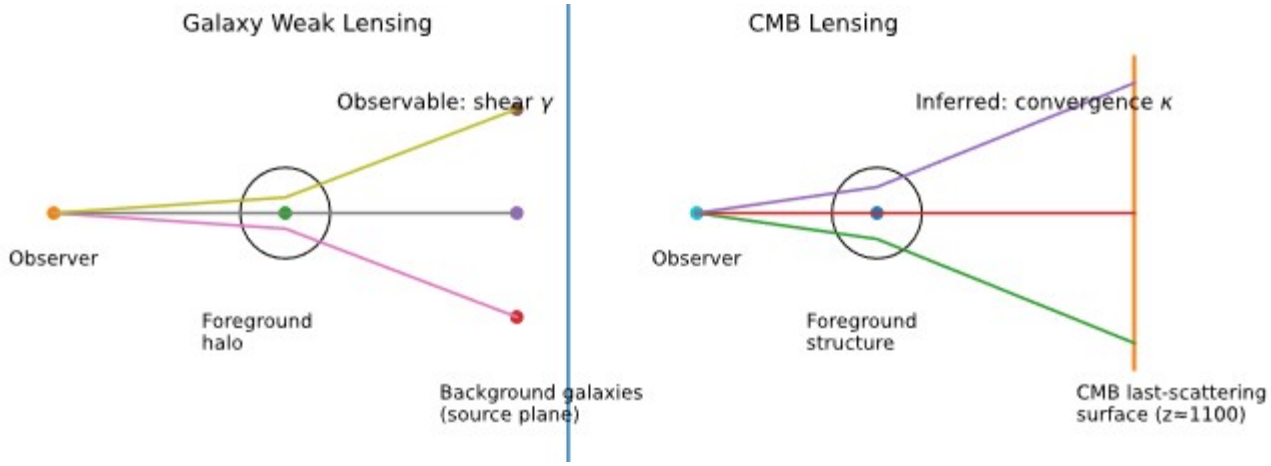
Weak-lensing example $C_\ell^{\kappa\kappa} \propto S_8^2$

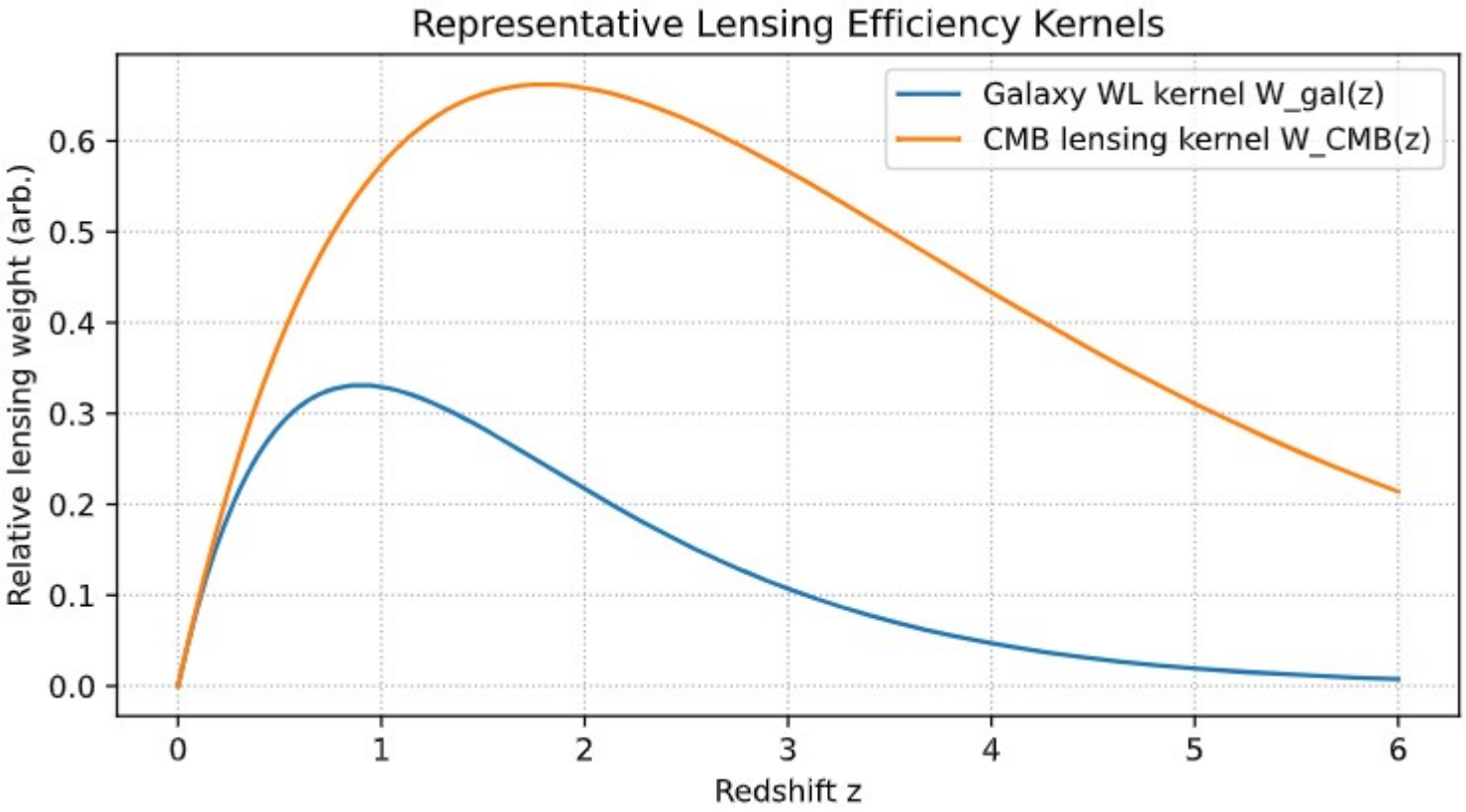
CMB example: $\theta_s = \dfrac{r_s(z_*)}{D_A(z_*)}$

What is labelled on the Weak Lensing Geometry

- *Observable:* **shear** γ (galaxy weak lensing)
- *Inferred:* **convergence** κ (mass reconstruction / CMB lensing)
- Source planes (galaxies vs CMB last-scattering surface)
- Foreground mass / halos
- Observer

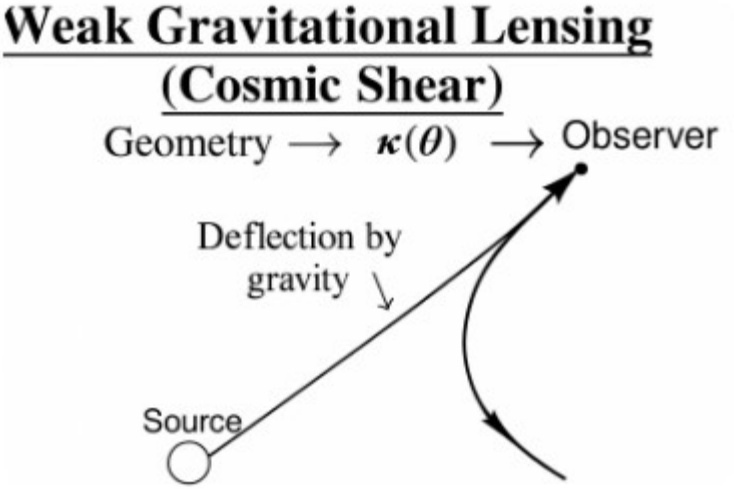
Weak Lensing Geometry





Weak Gravitational Lensing (Cosmic Shear)

Geometry → $\kappa(\theta)$ → C_l



Weak gravitational lensing constrains the combined parameter S_8

Making it a uniquely sensitive probe of structure growth
and a key diagnostic for testing ΛCDM and its extensions.

Connection to Modified Gravity and Dark Energy

Implications for New Physics

The sensitivity of weak gravitational lensing to S_8 makes it a powerful probe of physics beyond the standard ΛCDM model. Measurements of S_8 from weak-lensing surveys such as DES, KiDS, and HSC are systematically lower than the value inferred from the Cosmic Microwave Background under ΛCDM assumptions, giving rise to the so-called S_8 *tension*.

This tension may reflect:

- **Modified gravity**, in which the growth of structure deviates from General Relativity while leaving background expansion largely unchanged.
- **Evolving dark energy**, which alters the growth factor $D(z)$ and suppresses structure formation at late times.
- **Massive neutrinos**, which damp small-scale power and reduce the lensing signal.
- **Astrophysical systematics**, such as baryonic feedback or intrinsic alignments.

Because weak lensing directly probes the growth of structure rather than only the background geometry, it provides a crucial complement to CMB and supernova measurements. Any successful extension of ΛCDM must simultaneously account for both the expansion history and the observed value of S_8.

Python Program to Generate "Toy" Plots: *weak_lensing_kappa_power_spectrum.py*

Compute a toy weak-lensing convergence power spectrum assuming Eisenstein–Hu ***P(k)*** and a single source redshift.

κ is a weighted line-of-sight projection of the matter overdensity

$$\kappa(\boldsymbol{\theta}) = \frac{3H_0^2\Omega_m}{2c^2}\int_0^{\chi_s}\frac{(\chi_s-\chi)\chi}{\chi_s a(\chi)}\,\delta(\chi,\boldsymbol{\theta})\,d\chi$$

κ is a weighted line-of-sight projection of the matter overdensity

Depends on:

- Growth of structure
- Geometry via distances
- Scale factor a(χ)
- $H_0 \rightarrow$ geometry
- $S_8 \rightarrow$ growth

Weak gravitational lensing primarily constrains the parameter combination

$$S_8 = \sigma_8\cdot\sqrt{\frac{\Omega_m}{0.3}}$$

which captures the dominant degeneracy between the amplitude of matter fluctuations and the matter density.

$$\gamma(\ell) = e^{2i\phi}\,\kappa(\ell)$$

Weak lensing surveys measure shear, which is mathematically inverted to recover the projected mass field.
Weak lensing uniquely measures the projected matter density
without relying on luminous tracers or galaxy bias assumptions.

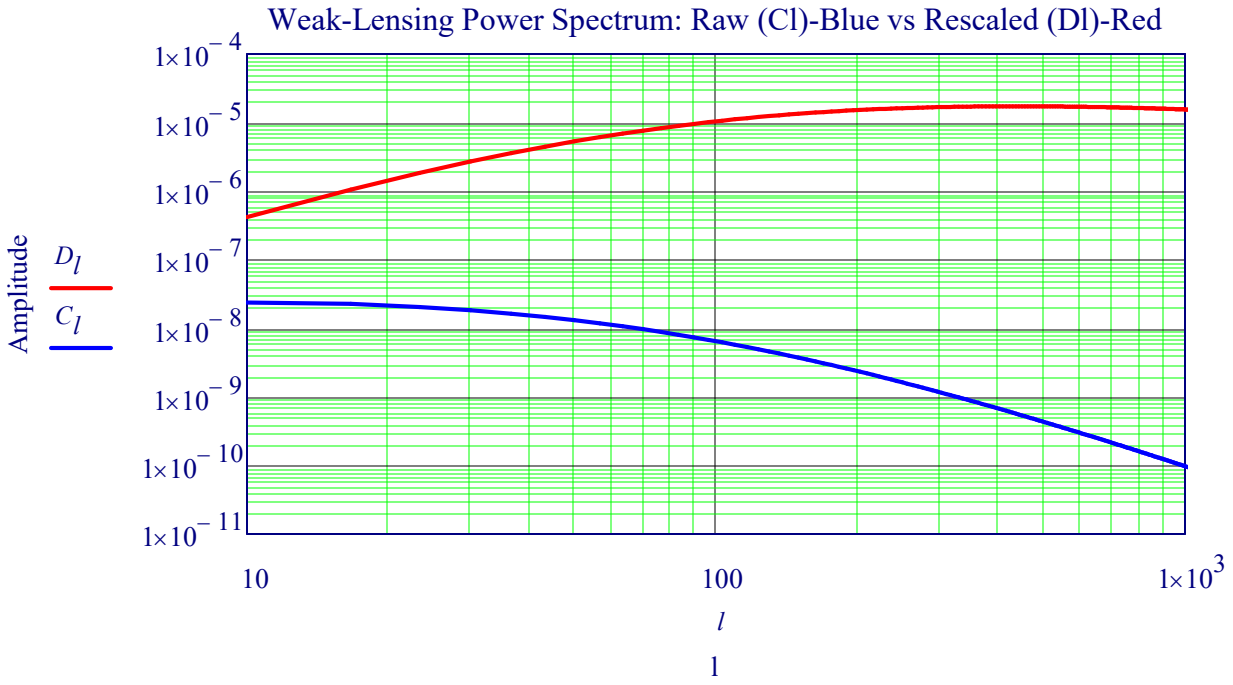
Toy weak-lensing convergence power spectrum

Flat ΛCDM: H_0=70.0 km/s/Mpc, $\Omega_m = 0.3$, $\Omega_L = 0.7$, $n_s = 0.965$, $\sigma_8 = 0.8$, $z_s = 1.0$

Data: l Cl $D_l = l(l+1)\,C_l/(2\pi)$

Read Data from Python Program: $WLens := READPRN("kappa_power_spectrum-Data2.txt")$

$l := WLens^{\langle 0\rangle}$ $C_l := WLens^{\langle 1\rangle}$ $D_l := WLens^{\langle 2\rangle}$



In the standard cosmological framework, weak gravitational lensing

encodes information about both the geometry of the Universe and the growth of structure. The convergence and shear fields depend on the matter power spectrum projected along the line of sight, making lensing sensitive to the total matter content and the amplitude of density fluctuations. Consequently, the lensing signal responds similarly to an increase in the matter density Ω_m or an increase in the fluctuation amplitude σ_8,

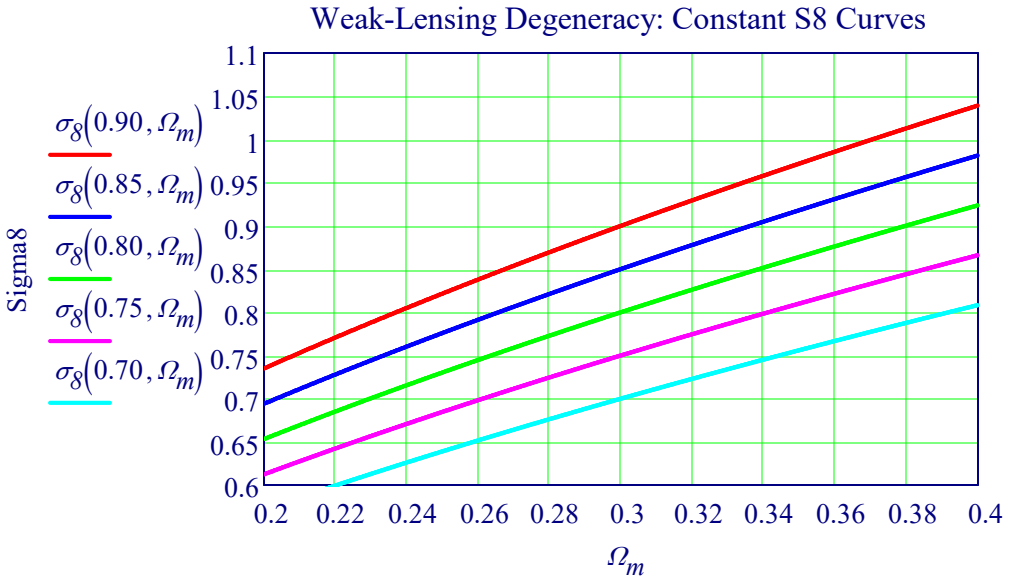
producing a degeneracy between these two parameters.

Rather than constraining Ω_m and σ_8 independently, weak-lensing data primarily constrain the combination

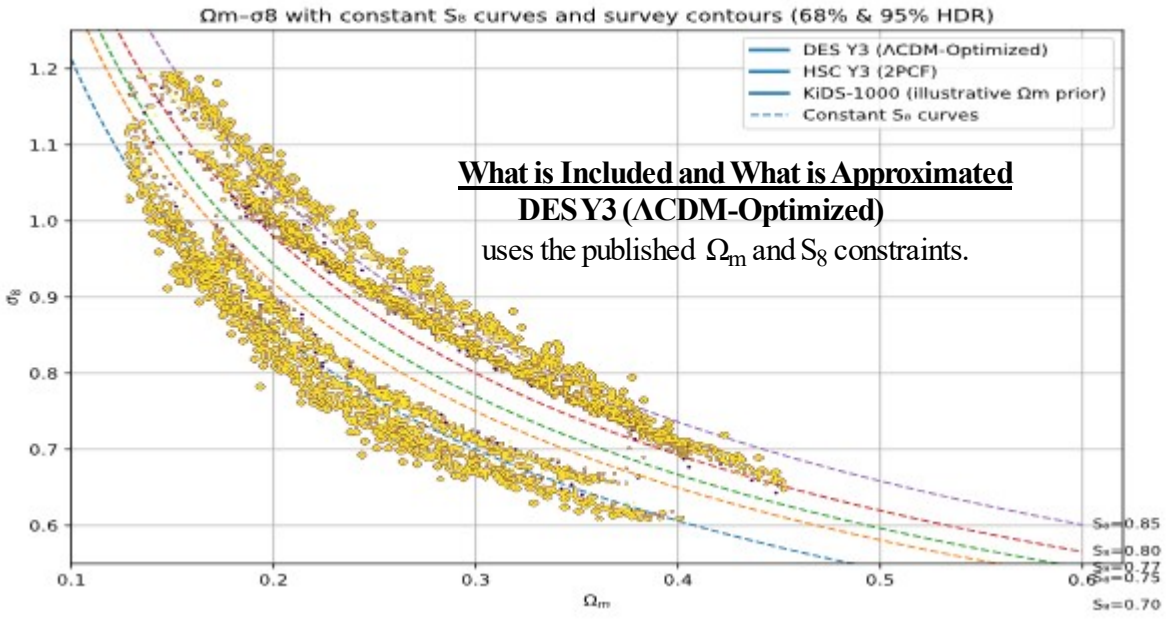
$$S_8 = \sigma_8 \sqrt{\Omega_m/0.3} \quad \Rightarrow \quad \sigma_8 = \frac{S_8}{\sqrt{\Omega_m/0.3}} \qquad \sigma_8(S_8, \Omega_m) := S_8 \cdot \left(\sqrt{\frac{\Omega_m}{0.3}}\right)$$

which traces the dominant direction of sensitivity in parameter space.

This behavior is illustrated in the following figure, where curves of constant S_8 define the characteristic degeneracy pattern observed in weak-lensing analyses.



Overlay DES / KiDS / HSC Likelihood Contours

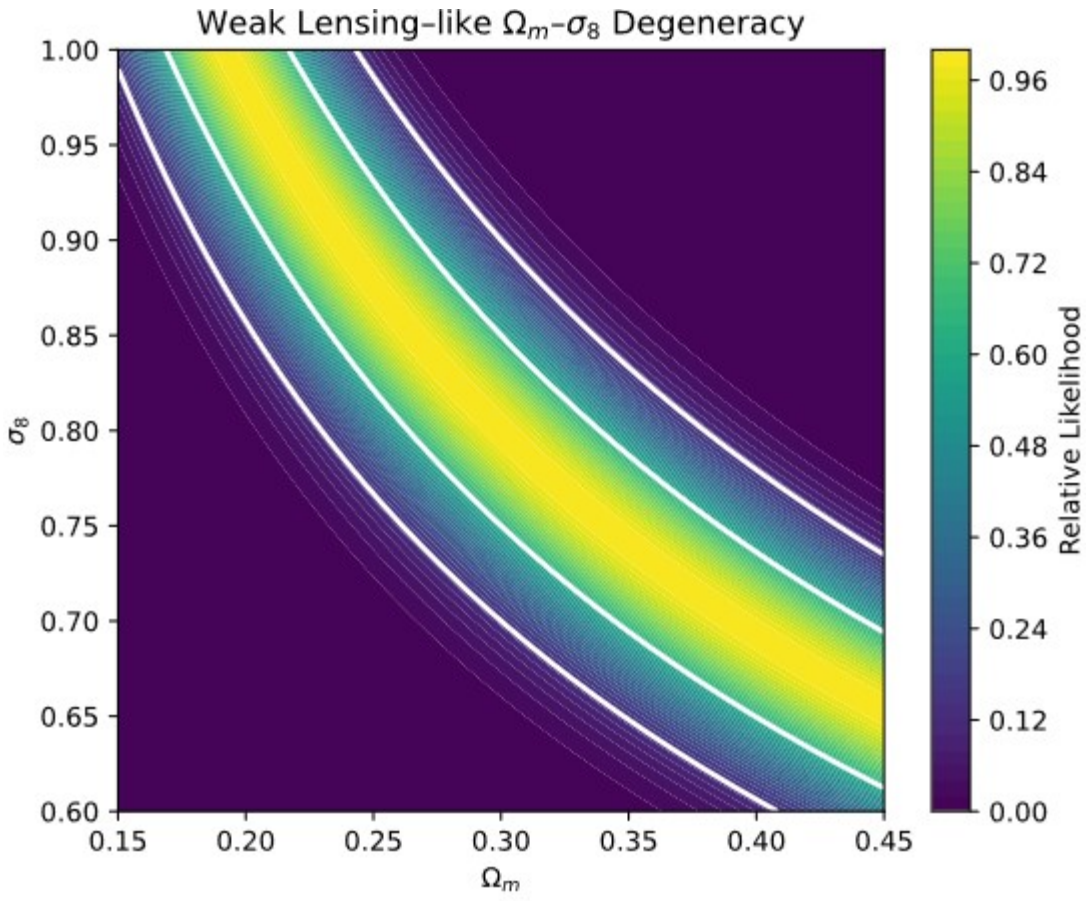


Degeneracy in Weak Gravitational Lensing

Weak gravitational lensing measurements probe the projected matter distribution through the coherent distortion of background galaxy shapes.
The amplitude of this signal depends on both the present-day normalization of matter fluctuations, σ_8, and the matter density parameter, Ω_m. As a result, cosmic shear surveys are most sensitive to the combined parameter

$$S_8 = \sigma_8 \cdot \sqrt{\frac{\Omega_m}{3}}$$

The Figure below illustrates the resulting degeneracy using an analytic likelihood constructed directly in the $\Omega_m - \sigma_8$ plane. The filled contours represent relative likelihood values, while the white curves indicate approximate 68% and 95% confidence regions.
The elongated shape of the contours reflects the fact that increases in Ωm can be partially compensated by decreases in σ_8, leaving the lensing signal approximately unchanged.
Although this figure is not derived from a full Markov-Chain Monte Carlo analysis, it reproduces the characteristic structure and orientation of weak-lensing constraints seen in surveys such as DES, KiDS, and HSC, making it useful for conceptual understanding and model comparison.



This figure is intended as a conceptual illustration;
full parameter constraints require detailed survey likelihoods, redshift distributions, and nonlinear modeling

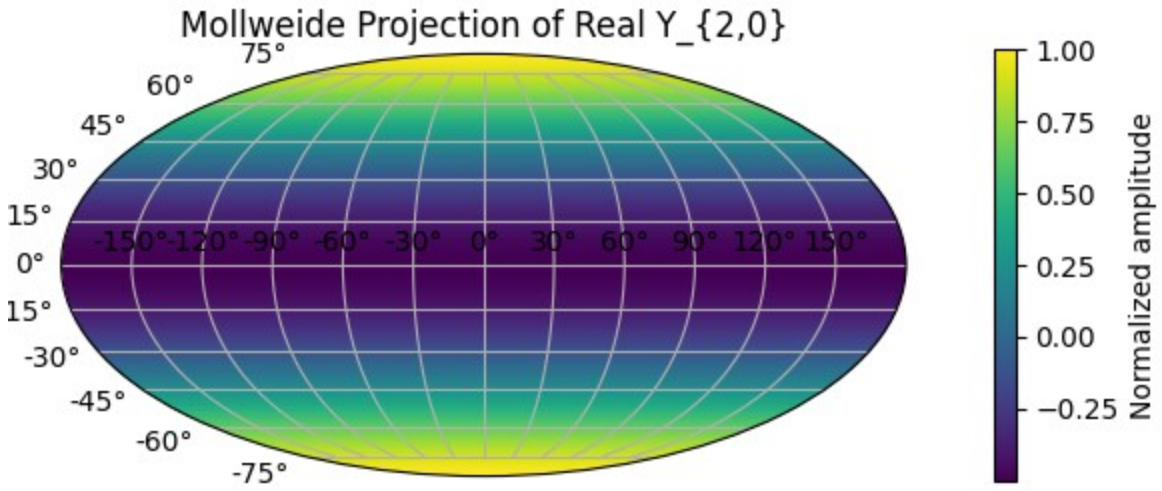
Mollweide Toy Multimode Map

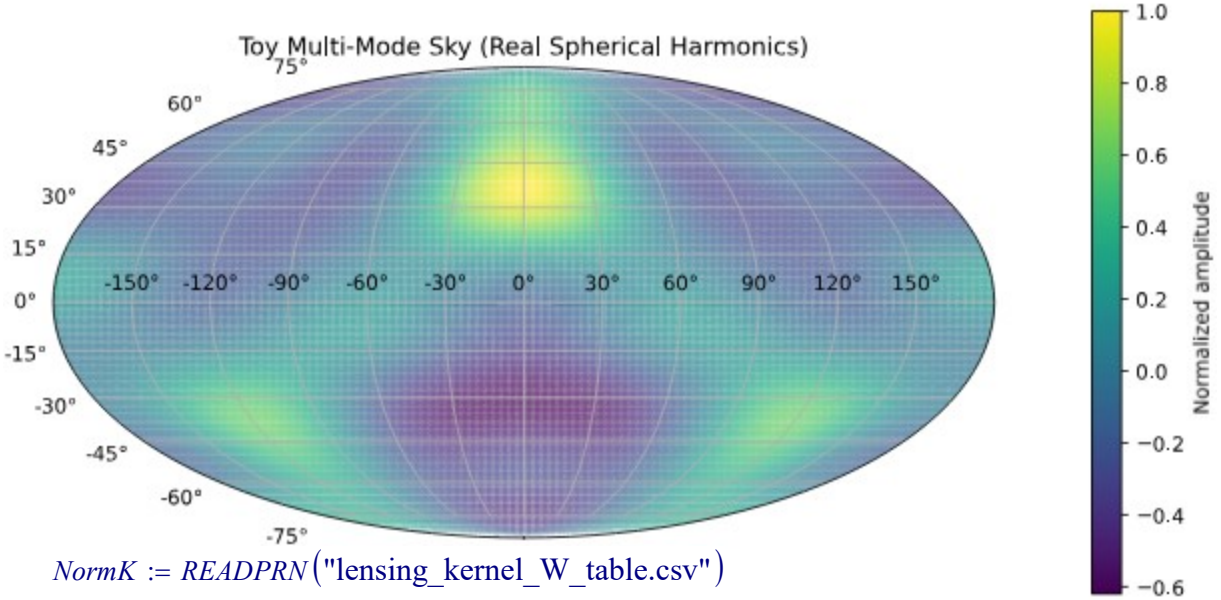
The Mollweide projection provides a **full-sky, equal-area representation of fields** defined on the celestial sphere and is widely used in cosmology to visualize Cosmic Microwave Background (CMB) temperature anisotropies, polarization patterns, and large-scale structure.
The figure mollweide_toy_multimode.svg presents a simplified, illustrative ("toy") sky map constructed from the superposition of multiple spherical harmonic–like modes. Rather than representing observational data, this visualization is designed to demonstrate how different angular scales and mode combinations manifest as coherent structures on the sphere when projected onto a two-dimensional map.

By combining low- and intermediate-order angular modes, the figure highlights key qualitative features such as large-scale hemispherical variations, nodal lines, and localized hot–cold regions that naturally arise in spherical decompositions. The Mollweide projection preserves relative area across the sky, allowing visual comparison of mode amplitudes and spatial correlations without distortion of surface weighting.

This toy multimode map serves as a pedagogical bridge between abstract harmonic coefficients $a_{\ell m}$ and their real-space sky patterns. It provides intuition for how power at different multipoles contributes to observed anisotropy maps and why full-sky projections are essential for interpreting angular correlations, power spectra, and symmetry properties in cosmological data analysis.

Uses the Python Program: mollweide_sphharm_and_lensing_kernel.py
Python Program is located: *VXPhysics.com/Python*

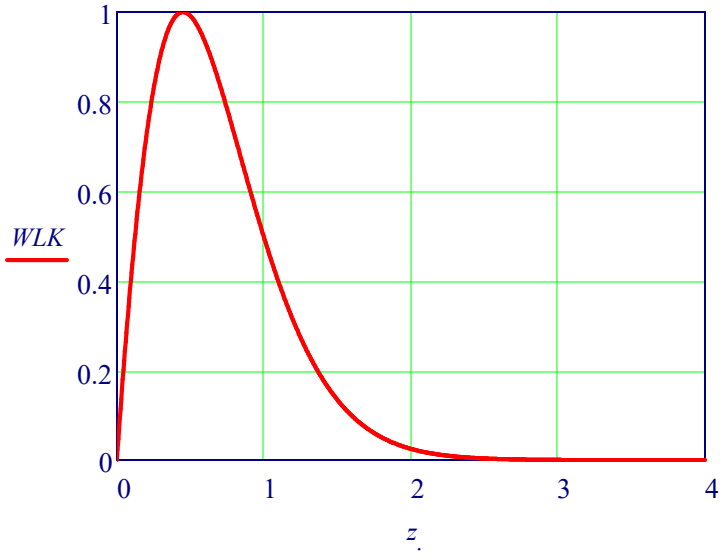




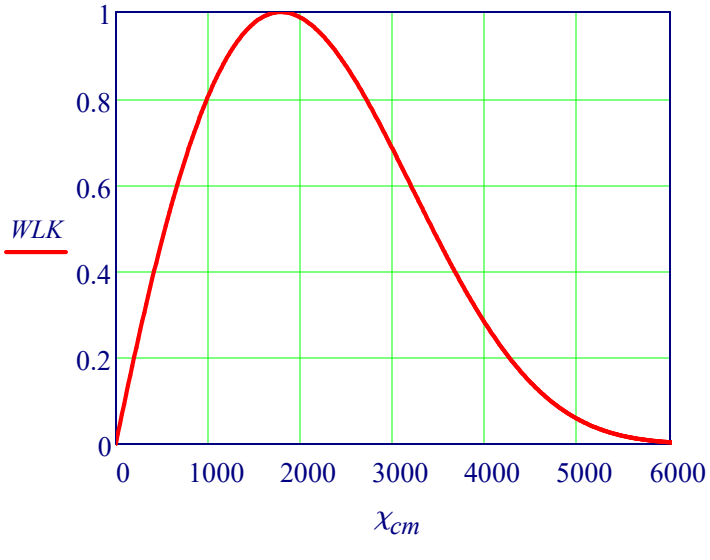
$NormK := READPRN("lensing_kernel_W_table.csv")$

$z := NormK^{\langle 0 \rangle}$ $\chi_{cm} := NormK^{\langle 1 \rangle}$ $WLK := NormK^{\langle 2 \rangle}$

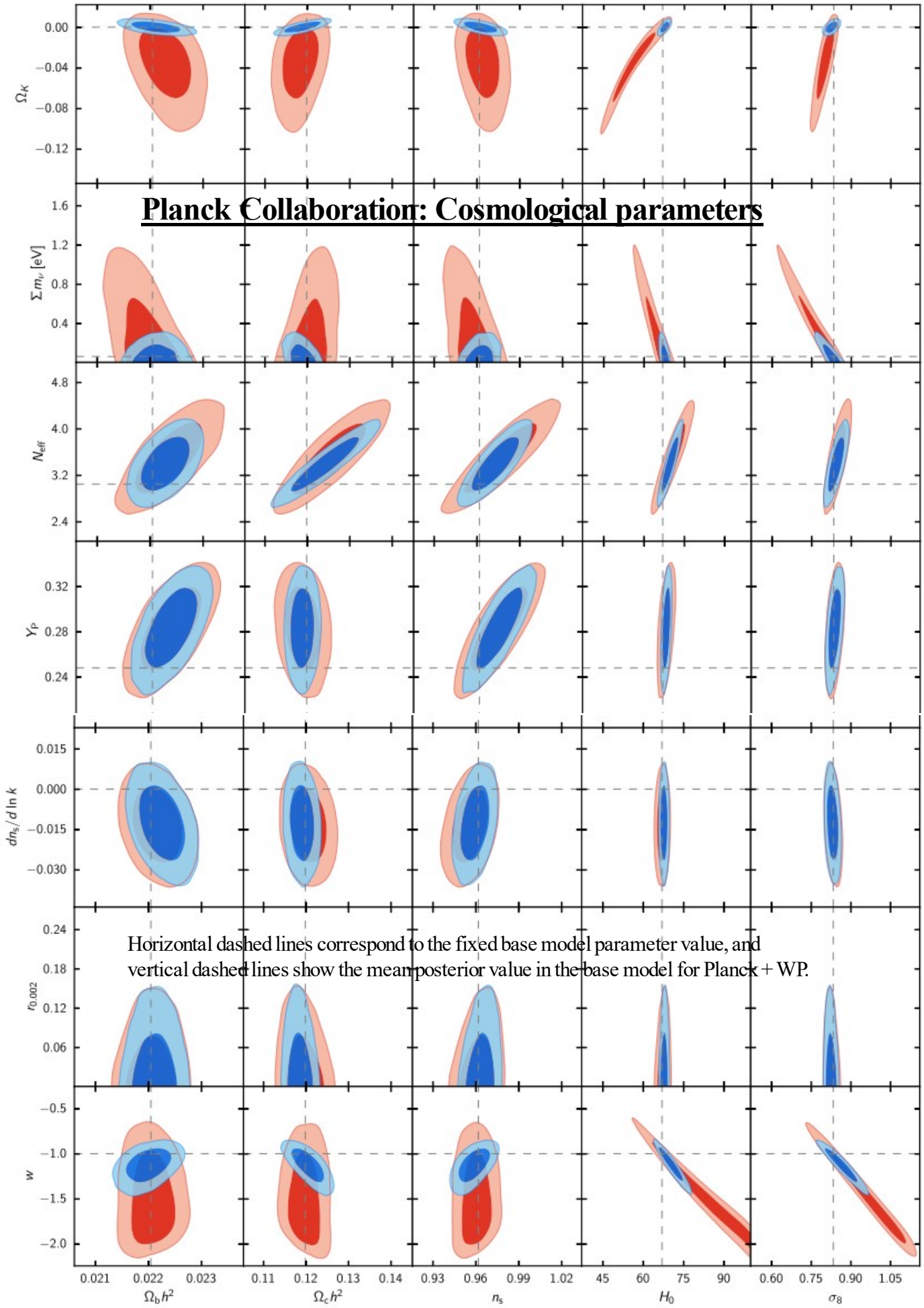
Radial Weak-Lensing Kernel vs Redshift, z



Radial Weak-Lensing Kernel vs x



Planck Collaboration: Cosmological parameters



Horizontal dashed lines correspond to the fixed base model parameter value, and vertical dashed lines show the mean posterior value in the base model for Planck + WP.

Uniformity of the CMBR is Evidence for Istropic Expansion and the ΛCDM

1998 COBE Far Infrared Absolute Spectrophotometer Monopole Spectrum Measurements

Assess If the Origin of the Cosmic Microwave Background Radiation (CMBR) is from the ΛCDM

COBE Measurements of CMBR Spectrum - Test: Surface of Last Scattering (from Clouds)? Thermal Blackbody?

Column 1 = Reciprocal Wavelength, λ, from Table 4 of Fixsen et al., in units = cm^{-1}

Column 2 = Intensity of FIRAS monopole spectrum computed as the sum of column 3, units = MJy/sr

$CMBR := READPRN("iras_monopole_spec_v1.txt")$ $T_{mw} := 2.7250K$

$\lambda := CMBR^{\langle 0 \rangle}$ $\lambda_6 = 4.99$ $I := CMBR^{\langle 1 \rangle}$ $n := 0, 1 .. rows(I) - 1$

$k_b := 1.3806505 \cdot 10^{-23} \cdot \frac{joule}{K}$ $h := 6.6260693 \cdot 10^{-34} \cdot joule \cdot sec$

Determine How Well COBE Spectrum Matches the Stretched Black Body Radiation at T = 2.750 K

Model: Equation for Intensity of Ideal Black Body Spectrum

$$B_\lambda(\lambda, T) := 2h \cdot c^2 \cdot \lambda^3 \cdot \left(e^{\frac{h \cdot c \cdot \lambda}{k_b \cdot T}} - 1 \right)^{-1}$$

Normalize Units at λ = 4.99

$$N_{unit} := I_6 \cdot B_\lambda\left(\frac{4.99}{cm}, T_{mw} \right)^{-1}$$

Wavelength of Light, λ, Stretches with Expansion

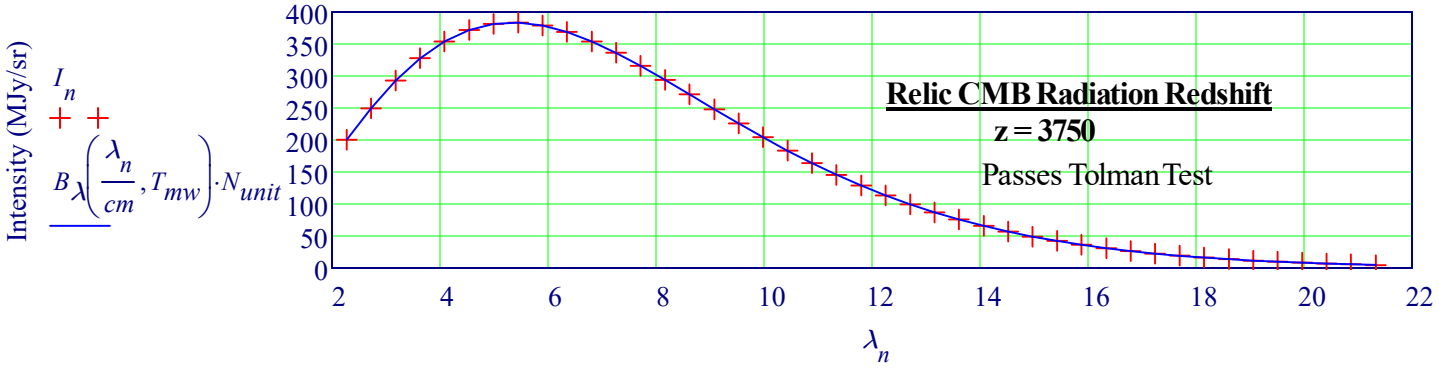
Stretches with the scale factor, a and λ stretch factor, z.

Given wavelength at emission, λ_o, λ today is

$$\lambda = \frac{1}{a(t)} \cdot \lambda_0 = (1 + z)\lambda_0$$

The CMBR has the most Perfect (Planckian) Black Body Spectral Curve known: T = 2.725 ± 0.001 K

Evidence: COBE CMB Radiation Black Body Spectrum is Perfect Fit for 2.725 K



Frequency Measured as Reciprocal Wavelength (1/cm)

CMB Energy: $N19 := 10^{-19}$ $eV := 1.6 \cdot 10^{-19} C \cdot volt = 1.6 J \cdot N19$ $k_b \cdot 2.75K = 0 \cdot eV$

Measured Uniformity (Low Error) of CMBR Temperature Reveals An Almost Perfect 2.725K Spectrum

$$Error\% := \frac{1}{rows(I) \cdot 100} \left[\sum_n \left(I_n - B_\lambda\left(\frac{\lambda_n}{cm}, T_{mw} \right) \cdot N_{unit} \right) \right]$$

$Error\% = 3.0064 \times 10^{-5}$

Scaling ==> $Temp(t) = \frac{T_0}{a(t)}$ $\lambda := \frac{c}{\nu} \approx$ **a**

CONCLUSION - ORIGIN OF CMBR:

The CMB radiation was emitted 13.7 billion years ago, only a few hundred thousand years after the ΛCDM, **long before stars or galaxies ever existed**. Radiation's temperature is defined by the wavelength of the individual photons that make it up. **As the Universe expands**, not only does the radiation get **less intense**, but the **stretching of space** will stretch the wavelength of the photons from the ΛCDM, which **decreases the energy** of the photons to longer wavelengths, which correspond to the energy of **lower temperatures**. When neutral atoms form, the radiation can no longer interact, and simply flies in a straight line until it interacts with something. **13.8 billion years later, that something is our eyes** and instruments, revealing an **ultra-cold, uniform bath** of radiation at 2.725 K. This is **Evidence of radiation from a hot, dense phase** in the past that many had theorized as representing the origin of our expanding Universe.

Λ-CDM Model Parameters

Wikipedia

"The current standard model of cosmology, the Lambda-CDM model, uses the FLRW metric. By combining the observation data from some experiments such as WMAP and Planck with theoretical results of Ehlers–Geren–Sachs theorem and its generalization, astrophysicists now agree that the early universe is almost homogeneous and isotropic (when averaged over a very large scale) and thus nearly a FLRW spacetime. That being said, attempts to confirm the purely kinematic interpretation of the Cosmic Microwave Background (CMB) dipole through studies of radio galaxies and quasars show disagreement in the magnitude. Taken at face value, these observations are at odds with the Universe being described by the FLRW metric. Moreover, one can argue that there is a maximum value to the Hubble constant within an FLRW cosmology tolerated by current observations, $H_0 = 73 \pm 8$ km/s/Mpc

Hubble Tension - Difference Between Local and Global Determinations of H_0

H_0 from **Cepheids and SNIa** = 73.04 $kms^{-1}Mpc^{-1}$ **Planck CMB** = 67.4 ± 0.5 $km^{s-1}Mpc^{-1}$. Discrepancy is $\approx 5\sigma$

2018 Planck CMB Results

https://arxiv.org/abs/1807.06209

Planck Collaboration 2018 Results. VI. Cosmological Parameters

	Description	Symbol	Value
Independent parameters	Baryon density today	$\Omega_b h^2$	0.0224±0.0001
	Cold dark matter density today	$\Omega_c h^2$	0.120±0.001
	100 × approximation to r∗/DA (CosmoMC)	$100\theta_{MC}$	1.04089±0.00031
	Reionization optical depth	τ	0.054±0.007
	Log power of the primordial curvature	$\ln(10^{10}A_s)$	3.043±0.014
	Scalar spectrum power-law index	n_s	0.965±0.004
Fixed parameters	Total matter density today (inc. massive neutrinos)	$\Omega_m h^2$	0.1428 ± 0.0011
	Equation of state of dark energy	w	$w_0 = -1$
	Tensor/scalar ratio	r	$r_{0.002} < 0.06$
	Running of spectral index	$dn_s/d\ln k$	0
	Sum of three neutrino masses	$\sum m_\nu$	0.06 eV/c^2
	Effective number of relativistic degrees of freedom	N_{eff}	2.99±0.17
	Hubble constant	H_0	67.4±0.5 $km{\cdot}s^{-1}{\cdot}Mpc^{-1}$
Calculated Values	Age of the universe	t_0	$(13.787)\times10^9$ years
	Dark energy density parameter	Ω_Λ	0.6847±0.0073
	The present root-mean-square matter fluctuation, averaged over a sphere of radius $8h^{-1}$ Mpc	σ_8	0.811±0.006
	Redshift of reionization (with uniform prior)	z_{re}	7.68±0.79

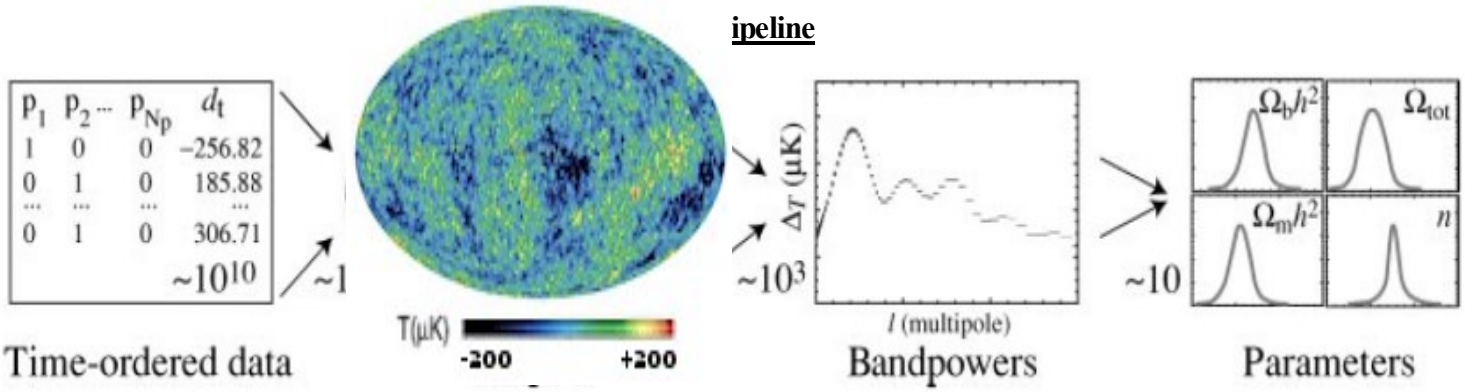
CMB Data Analysis Methodology: Angular Temperature Power Spectrum (TT)

CMB Data Analysis Methodology

Data pipeline and radical compression. Maps are constructed for each frequency channel from the data timestreams, combined, and cleaned of foreground contamination by spatial (represented here by excising the galaxy) and frequency information. Bandpowers are extracted from the maps and cosmological parameters from the bandpowers. Each step involves a substantial reduction in the number of parameters needed to describe the data, from potentially 10^{10} —> 10 for the Planck satellite.

In every step of CMB data analysis the aim is to reduce the volume of data without losing information.

ipeline



CMB temperature anisotropies are expressed in terms of multipoles:

If δT/T is expanded in terms of **Spherical Harmonics:** Y_{lm}

$$\frac{\delta T(\theta,\phi)}{T_0} = \sum_{l,m} a_{lm} Y_{lm}(\theta,\ \phi)$$

then the complex coefficients a_{lm}, in a homogeneous and isotropic universe, satisfy the condition

$$a_{lm} = \int \frac{\Delta T(n)}{T} \cdot Y_{lm}(n)\ dn$$

$$Y_0^0(\theta,\phi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

$$Y_1^{-1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{-i\phi}$$

$$Y_1^0(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$$

$$Y_1^1(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{i\phi}$$

It is the **variance of the temperature field** which **carries the cosmological information**, rather than the values of the individual $a_{\ell m}$s; in other words the power spectrum in ℓ fully characterizes the anisotropies. The power at each ℓ is $(2\ell+1)C_\ell/(4\pi)$, and a statistically isotropic sky means that all ms are equivalent.

Where a_{lm} follow the Gaussian (maximally randomized) distribution with zero mean and variance given by C_l :

$$\langle a^*_{l'm'} a_{lm}\rangle = \delta_{ll'}\,\delta_{mm'}\,C_l \qquad C_\ell \equiv \left\langle |a_{\ell m}|^2 \right\rangle$$

An unbiased estimator of C_l is defined as:

$$C_l = \frac{1}{2l+1}\cdot \sum_{m=-l}^{l} \left(a_{lm}\cdot a_{lm}\right)^*$$

CMB Likelihood

- CMB temperature and polarization observations can constrain cosmological parameters if the likelihood function can be computed exactly.
- Computing the likelihood function exactly in a brute force way is computationally challenging since it involves inversion of the covariance matrix i.e., $O(N^3)$ computation.
- In Cosmological parameter estimation a theoretical model is represented by its angular power spectrum C_l .
- For a set cosmological parameters we can compute the angular power spectrum C_l using publicly available Boltzmann codes like **CMBFAST** and **CAMB** (**Code for Anisotropies in the Microwave Background**) and try to fit that with observed C_l . **CMBquick** (Refer to Section XVI) is implemented in Mathematica.