XXXI. Three Analyses of the Flatness Problem - The Fine Tuning Problem

A. *An Introduction To Modern Cosmology,* Andrew Liddle
B. *A survey of dark matter and related topics in cosmology,* Bing-Lin Young, Phys. 12(2), 121201 (2017),
C. *Astronomy 275 Lecture Notes,* Edward Wright: Spring 2015, Section 6.1

What is the Flatness Problem?

Recent measurements of the total density of the Universe find $0.95 < \Omega o < 1.05$. This near flatness is a problem because the Friedmann Equation tells us that $\Omega \approx 1$ is a very unstable condition - like a pencil balancing on its point. It is a very special condition that won't stay there long. Here is an example of how special it is. We know that

$(\Omega^{-1} - 1)\,\rho R^2 =$ constant. Therefore, we can write,

$$\left(\Omega^{-1} - 1\right)\rho \cdot R^2 = \left(\Omega_0^{-1} - 1\right)\rho_0 \cdot R_0^2$$

where the right hand side is today and the left hand side is at any arbitrary time. We then have,

$$\left(\Omega^{-1} - 1\right) = \left(\Omega_0^{-1} - 1\right)\frac{\rho_0}{\rho} \cdot \left(\frac{R_0}{R}\right)^2$$

redshift is related to the scale factor by $R = Ro/(1 + z)$. Consider **the evolution during matter-domination** where $\rho = \rho_o(1 + z)^3$. Inserting these we get,

$$\left(\Omega^{-1} - 1\right) = \frac{\left(\Omega_0^{-1} - 1\right)}{1 + z}$$

Inserting the current limits on the density of the Universe, **$0.95 < \Omega o < 1.05$**

(for which $-0.05 < (\Omega o^{-1} - 1) < 0.05$),

we get a constraint on the possible values that could have had at redshift, z.

$$\frac{1}{1 + \frac{0.05}{1 + z}} < \Omega < \frac{1}{1 - \frac{0.05}{1 + z}} \qquad \Delta\Omega(z, \phi) := \frac{1}{1 - \frac{\phi \cdot 0.05}{1 + z}}$$

At recombination (when the first hydrogen atoms were formed) $z \approx 10^3$ and the constraint on Ω yields,

$$\Delta\Omega(1000, -1) = 0.99995 \quad < \Omega < \quad \Delta\Omega(1000, 1) = 1.00005$$

So the observation that $0.95 < \Omega o < 1.05$ today, means that at a redshift of $z \approx 10^3$ we must have had $0.99995 < \Omega < 1.000005$. This range is small... special. However, had to be even more special earlier on. We know that the standard ΛCDM successfully predicts the relative abundances of the light nuclei during nucleosynthesis between ≈ 1 minute and ≈ 3 minutes after the big bang, so let's consider the slightly earlier time, 1 second after the big bang which is about the beginning of the epoch in which we are confident that the Friedmann Equation holds. The redshift was $z \approx 10^{11}$ and the resulting constraint on the density at that time was,

$$\Delta\Omega\left(1 \cdot 10^{11}, -1\right) = 0.9999999999995 \quad < \Omega < \quad \Delta\Omega\left(1 \cdot 10^{11}, 1\right) = 1.0000000000005$$

This range is even smaller and more special, (although we have **assumed matter domination** for this calculation, at redshifts higher than $z_{eq} \approx 3000$, we have **radiation domination** and $\rho = \rho_o(1 + z)^4$. This makes the $1 + z$ in the equation a $(1 + z)^2$ and requires that early values of be even closer to 1 than calculated here).

To summarize:

$$0.95 < \Omega_o(z = 0) < 1.05$$
$$0.99995 < \Omega(z = 10^3) < 1.000005$$
$$0.9999999999995 < \Omega(z = 10^{11}) < 1.0000000000005$$

A. *An Introduction To Modern Cosmology,* Andrew Liddle

The flatness problem is the easiest one to understand. We have learned that the Universe possesses a total density of material, $\Omega_{tot} = \Omega_0 + \Omega_A$, which is close to the critical density. Very conservatively, it is known to lie in the range $0.5 < \Omega_{tot} < 1.5$. In terms of geometry, that means that the Universe is quite close to possessing the flat (Euclidean) geometry. We have seen that the Friedmann equation can be rewritten as an equation showing how Ω_{tot} varies with time. Adding modulus signs to the Friedmann Equation gives:

$$|\Omega_{\rm tot}(t) - 1| = \frac{|k|}{a^2 H^2}$$

We know from this that this that $\boldsymbol{\Omega_{tot}}$ is precisely equal to one, then it remains so for all time. But what if it is not?

Let's consider the situation where we have a conventional Universe (matter or radiation dominated) where the normal matter is more important than the curvature or cosmological constant term. Then we can use the solutions ignoring the curvature term, using equations to find

$a^2H^2 \propto t^{-1}$	radiation domination	$\lvert\Omega_{\rm tot} - 1\rvert \propto t$	radiation domination
$a^2H^2 \propto t^{-2/3}$	matter domination .	$\lvert\Omega_{\rm tot} - 1\rvert \propto t^{2/3}$	matter domination .

In either case, the difference between Ω_{tot} and 1 is an *increasing* function of time. That means that the flat geometry is an unstable situation for the Universe; if there is any deviation from it then the Universe will very quickly become more and more curved. Consequently, for the Universe to be so close to flat even at its large present age means that at very early times it must have been extremely close to the flat geometry.

An alternative way to see this is to remember that the densities of matter and radiation reduce with expansion as l/a^3 and l/a^4 respectively. These are both faster reductions than the curvature term k/a^2. So if the curvature term is not to totally dominate in the present Universe, it must have begun much smaller than the other terms.

The equations for $|\Omega_{tot} - 1|$ derived above stop being valid once the curvature or cosmological constant terms are no longer negligible, since we used the *a(t)* solutions for the flat geometry to derive them. But they are fine to give us an approximate idea of what the problem is. For extra ease let's assume that the Universe always has only radiation in it. Using the equations above, we can ask how close to one the density parameter must have been at various early times, based on the constraint today ($t_o \approx 4 \times 10^{17}$ sec).

- At decoupling ($t \simeq 10^{13}$ sec), we need $|\Omega_{\rm tot} - 1| \leq 10^{-5}$.
- At matter–radiation equality ($t \simeq 10^{12}$ sec), we need $|\Omega_{\rm tot} - 1| \leq 10^{-6}$.
- At nucleosynthesis ($t \simeq 1$ sec), we need $|\Omega_{\rm tot} - 1| \leq 10^{-18}$.
- At the scale of electro-weak symmetry breaking, which corresponds to the earliest known physics ($t \simeq 10^{-12}$ sec), we need $|\Omega_{\rm tot} - 1| \leq 10^{-30}$.

Written out in long hand, that means we know that at nucleosynthesis, an era we are supposed to understand very well indeed, the density parameter must have lain within the range $0.999999999999999999 < \Omega_{tot} < 1.000000000000000001$! Out of all the possible values that it might have had, this seems a very restrictive range. Any other value would lead to a Universe extremely different to that which we see.

The easiest way out of this dilemma is to suppose that the Universe must have precisely the critical density. But on the face of it there seems no reason to prefer this choice over any other. What would be nice would be an explanation of such a value.

Regardless of whether or not we understand the physical origin of these numbers. they are an observed fact. One useful thing they tell us is that the Universe is very close to spatial flatness at decoupling and at nucleosynthesis, which means that it is always a good approximation to set k = 0 in the Friedmann equation when describing those phenomena.

B. *A survey of dark matter and related topics in cosmology*

Phys. 12(2), 121201 (2017), Bing-Lin Young

We note that in the very early universe **radiation energy dominates**. Then $H^2(z) \approx (1+z)^4$ which says

$$\Omega_k(z) \approx (1+z)^{-2}\,\Omega_k$$

This gives rise to the well-known flatness problem. For any finite value of the curvature parameter, i.e., any value of Ω_k at the present epoch, the curvature fraction to the effective total energy density is negligibly small at the early universe of $z \gg 1$. Running the argument in the reversed direction with $\Omega_k \approx (1+z)^2\,\Omega_k(z)$, we have

a z^2 growth in the curvature density fraction.

From the fact that the observed matter-energy density today ρ_0 is close to the critical density ρ_c, this requires a very small curvature density fraction in the early universe. This gives raise to a fine tuning problem unless $k = 0$: Furthermore, a finite curvature constant allows the determination of the scale factor at the present time, a_0, **which is unphysical**, from the equation

$$\rho_\kappa = \frac{3}{8\pi G_N}\,\frac{\kappa c^2}{a_o^2} \qquad D_H := \frac{c}{H_0}$$

Critical Density:

$$\Omega_{crit}(h) := 7.5\cdot 10^{21}\,h^{-1}\cdot M_{\odot}\,D_H^{-3}$$

C. Astronomy 275 Lecture Notes: Edward Wright

Edward Wright: Spring 2015, Section 6.1.The Flatness-Oldness Figure 14 (https://astro.ucla.edu/≈wright/)

The expanding universe evolves away from $\Omega_{tot} = 1$:

Note: See the Following Page for more details

***ΩkHR* Equation:**

$$1-\Omega(t) = -\frac{kc^2}{H(t)^2 a(t)^2 R_0^2} = \frac{H_0^2(1-\Omega_0)}{H(t)^2 a(t)^2}$$

$$\left(\frac{H(t)}{H_0}\right)^2 = \frac{\Omega_{r0}}{a^4} + \frac{\Omega_{m0}}{a^3}$$

This creates an enormous fine-tuning problem:

the early universe must have been remarkably close to $\Omega_{tot} = 1$ in order to have $\Omega_{tot} \approx 1$ today!

Just 1 gm/cc out of $447*10^{21}$- gm/cc at 1 ns is the difference between an expanding, flat, or closed universe.