XXXIII Proof of the Borde-Guth-Vilenkin (BGV) Theorem

The beginning of-the universe.
The Borde Guth Vilenkin Theorem, indefinitely continued into past., Vilenkin,
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The BGV theorem demonstrates "that **any inflating model that is globally expanding must be geodesically incomplete in the past**".

Was the big bang truly the beginning of the universe? A beginning in what? Caused by what? And determined by what, or whom? These questions have prompted physicists to make every attempt to avoid a cosmic beginning.

Physicists hoped initially that the singularity might be an artifact of Friedmann's simplifying assumption of perfect uniformity, and that it would disappear in more realistic solutions of Einstein's equations. Roger Penrose closed this loophole in the mid-1960s by showing that, under a very general assumption, the singularity was unavoidable. Under the null convergence condition, gravity always forces light rays to converge.
(Mathematically, t**he null convergence condition (NCC) requires** that the Ricci curvature tensor $R_{\mu\nu}$ must satisfy $R_{\mu\nu}N^{\mu}N^{\nu} \geq 0$ for all null vectors N^{μ}. A null vector is a vector of zero norm, $N\mu N\mu=0$. Combined with Einstein's equations, NCC is equivalent to the null energy condition (NEC), requiring that $T_{\mu\nu}N_{\mu}N_{\nu} \geq 0$ for all null N_{μ}, where $T_{\mu\nu}$ is the Einstein Energy-Momentum Tensor.)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{M_{\mathrm{pl}}^2}T_{\mu\nu}$$

Proof

Start with a homogeneous, isotropic, and spatially flat universe with the metric:
This implies that the **density of matter or energy measured by any observer cannot be negative**. The conclusion holds for all familiar forms of classical matter.

$$ds = dt^2 - a^2(t)dx_i dx^i$$

The Hubble expansion rate is $H = \dot{a}/a$, where the dot denotes a derivative with respect to time t. We can imagine that the universe is filled with comoving particles, moving along the timelike geodesics vector x = const. Consider an inertial observer, whose world line is $x_{\mu}(\tau)$, parametrized by the proper time τ. For an observer of mass m, the 4-momentum is $P^{\mu}=m\, dx^{\mu}/d\tau$, so that $d\tau = (m/E)dt$ where $E = P^0 = (p_2+m_2)^{1/2}$ denotes the energy, and p, the magnitude of the 3-momentum. It follows from the geodesic equation of motion that $p \propto 1/a(t)$, so that

$p(t) = [a(t_f)/a(t)]p_f$, where p_f designates the momentum at some reference time t_f.

Thus
$$\int_{t_i}^{t_f} H(\tau)d\tau = \int_{a(t_i)}^{a(t_f)} \frac{m\,da}{\sqrt{m^2a^2 + p^2a(t_f)}} = F(\gamma_f) - F(\gamma_i) \leq F(\gamma_f)$$

where $t_i < t_f$ is some initial moment.
Note that:

$$F(\gamma) = \frac{1}{2}\ln\left(\frac{\gamma+1}{\gamma-1}\right) \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\nu^2}}$$

γ is the Lorentz factor, and $v_{rel} = p/E$ is the observer's speed relative to the comoving particles.

For any non-comoving observer, $\gamma > 1$ and $F(\gamma) > 0$

The expansion rate averaged over the observer world line can be defined as

$$\text{Define: } H_{av} = \frac{1}{\tau_f - \tau_i} \int_{t_i}^{\tau_f} H(\tau) d\tau.$$

Assuming that $H_{av} > 0$ and using the first equation, it follows that

$$\tau_f - \tau_i \leq \frac{F(\gamma_f)}{H_{av}}.$$

This implies that any non-comoving past-directed timelike geodesic satisfying the condition $H_{av} > 0$, must have a finite proper length, and so must be past-incomplete.

There is no appealing to homogeneity and isotropy in an arbitrary space-time. Imagine that the universe is filled with a congruence of comoving geodesics, representing test particles and consider a non-comoving geodesic observer described by a world line $x_\mu(\tau)$

Let u_μ and ν^μ designate the 4-velocities of test particles and the observer.
Then the Lorentz factor of the observer relative to the particles is

$$\gamma = u_\mu \nu^\mu$$

To characterize the expansion rate in general space-time, it suffices to focus on test particle geodesics that cross the observer's world line. Consider two such geodesics encountering the observer at times τ and $\tau + \Delta\tau$.
Define the parameter

$$H = \frac{d}{d\tau} F(\gamma(\tau))$$

with $F(\gamma) = 1/\gamma$, and γ defined by

$$H = \lim_{\Delta\tau \to 0} \frac{\Delta u_r}{\Delta r}$$

Clearly, $F(\gamma) > 0$, and the argument goes through as before.

In general relativity, a timelike congruence in a four-dimensional Lorentzian manifold can be interpreted as a family of world lines of certain ideal observers in our spacetime.

A rigorous formulation of the BGV theorem is now possible.
Let λ be a timelike or null geodesic maximally extended to the past, and let C be a timelike geodesic congruence defined along λ.

A universe that has been expanding on average throughout its history cannot be infinite in the past but must have a beginning.

If the expansion rate of $\mathcal{C}$ averaged along λ is positive, then λ must be past-incomplete.