Space Time Theorems

I. Theorems About Cosmic Origins, Singularities, and Time

1. Hawking–Penrose Singularity Theorems (1965–1970)

Status: Foundational (more fundamental than BGV)

What they say:

Under very general physical conditions, including reasonable energy conditions and causal structure, spacetime must be geodesically incomplete. In cosmological and gravitational-collapse scenarios, this incompleteness is interpreted as the presence of spacetime singularities.

Why they matter:

These theorems provided the first rigorous demonstration that general relativity predicts its own breakdown under physically realistic conditions. They apply directly to Big Bang cosmology and black hole formation and do not rely on inflationary assumptions.

Relation to BGV:

The Hawking–Penrose theorems rely on stronger energy conditions but are narrower in scope. By contrast, BGV weakens the assumptions while extending applicability to inflationary spacetimes.

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2. Borde–Guth–Vilenkin (BGV) Theorem (2003)

What it says:

Any universe that has, on average, a positive Hubble expansion along a past-directed timelike or null geodesic is geodesically past-incomplete, regardless of energy conditions.

Why it matters:

This result applies even when the classical energy conditions required by the Hawking–Penrose theorems fail. It encompasses eternal inflation, cyclic models, and emergent-universe scenarios, and it does not assume Einstein's field equations.

Key distinction:

The BGV theorem is kinematic rather than dynamical, relying only on spacetime expansion properties rather than gravitational field equations.

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3. Raychaudhuri Equation (Theorem-like Role)

Status: Structural backbone of singularity proofs

What it says:

The Raychaudhuri equation governs the evolution of geodesic congruences and shows that, under attractive gravity, initially converging geodesics inevitably focus, leading to caustics and geodesic incompleteness.

Why it matters:

This equation underlies all modern singularity theorems. It connects expansion, shear, vorticity, and energy conditions, and explains why singularities arise generically rather than as exceptional solutions.

In this work:

The Raychaudhuri equation serves as the mathematical engine behind both the Hawking–Penrose and BGV theorems.

II. Theorems That Constrain Cosmological Models

4. Cosmic No-Hair Theorem

What it says:

Universes with a positive cosmological constant evolve toward de Sitter–like states, dynamically suppressing initial anisotropies and inhomogeneities.

Why it matters:

This theorem explains the observed large-scale isotropy of the universe and provides theoretical support for both inflationary dynamics and late-time cosmic acceleration.

Connection to BGV:

The Cosmic No-Hair theorem explains why inflation works locally even if inflationary spacetimes remain past-incomplete globally, as indicated by BGV.

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5. Ehlers–Geren–Sachs (EGS) Theorem

What it says:

If all fundamental observers measure an exactly isotropic cosmic microwave background, then the spacetime geometry must be Friedmann–Lemaître–Robertson–Walker (FLRW), under mild physical assumptions.

Why it matters:

This theorem provides a direct observational justification for the cosmological principle and links CMB isotropy to large-scale spacetime geometry.

Role in cosmology:

EGS elevates the use of FLRW geometry from a philosophical assumption to an empirical inference.

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6. Almost–Ehlers–Geren–Sachs (Almost-EGS) Theorem

What it says:

If the cosmic microwave background is nearly isotropic for all observers, then the universe must be close to an FLRW spacetime.

Why it matters:

This theorem extends EGS to realistic observational conditions, where small anisotropies are present, and justifies perturbative treatments of cosmological structure.

Role in cosmology:

Almost-EGS bridges idealized exact results with real CMB measurements.

III. Theorems About Perturbations and Structure Formation

7. Sachs–Wolfe Theorem

What it says:

Gravitational potential fluctuations at the surface of last scattering produce temperature anisotropies in the cosmic microwave background on large angular scales.

Why it matters:

This theorem establishes a direct link between general-relativistic perturbations and observable CMB anisotropies and explains the low-multipole plateau in the CMB temperature power spectrum.

Cosmological significance:

It connects primordial curvature perturbations to late-time observables.

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8. Weinberg Adiabatic Mode Theorem

What it says:

Certain long-wavelength adiabatic perturbation modes remain conserved outside the cosmological horizon.

Why it matters:

This conservation law justifies extrapolating primordial perturbations from inflationary epochs to recombination and beyond.

Role in cosmology:

The theorem underpins the robustness of inflationary predictions and the near scale-invariance of large-scale structure.

IV. Theorems About Expansion and Distance Measures

9. Etherington Reciprocity Theorem (Distance Duality)

What it says:

In any metric theory of gravity where photon number is conserved, the luminosity distance D_L and angular-diameter distance D_A are related by

$D_L = (1+z)^2 D_A$.

Why it matters:

This theorem links independent cosmological distance measures and provides a powerful observational consistency test.

Observational relevance:

Violations would signal exotic physics such as photon non-conservation or non-metric gravity.

10. Friedmann Theorem (FLRW Uniqueness)

What it says:

Assuming large-scale homogeneity and isotropy, the FLRW metric is the unique solution of Einstein's field equations.