I. Simple Lunar Trajectories: Kepler's Elliptical Model (Planar Point Mass)

This Section on Kepler is shown for historical interest. Newton's Dynamics is used in all the following Sections

Kepler's E Model (Planar Point Mass 2 Body): See the **Glossary** and **Figures** in **last two pages** of this Study

Convert Cartesian Ellipse Eq. in (x,y) to polar (r,v) coordinates $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ $\quad r(\nu, e) := \frac{a\cdot(1-e^2)}{1 + e\cdot\cos\left(\nu - \frac{\pi}{2}\right)}$

Ellipse is relative to the **focus**

$x(a,\theta) := a\cdot\cos(\theta)$ and $y(b,\theta) := b\cdot\sin(\theta)$

$0 \le t < 2\pi$ $\quad e = \frac{c}{a}$ $\quad r(x,y) := \sqrt{x^2+y^2}$ and $\theta(x,y) := \text{atan}\left(\frac{y}{x}\right)$

For the moon

$e_m := .0549$ $\quad d_m := 384400\text{km}$ $\quad d_{ap} := 406603\text{km}$ $\quad m_m := 7.347\cdot10^{22}\text{kg}$ $\quad a_m := \frac{d_{ap}}{1+e}$ $\quad \mu := 3.986\cdot10^5\frac{\text{km}^3}{\text{sec}^2}$

For the Earth: Mass $m_e := 5.972\cdot10^{24}\text{kg}$ $\qquad$ Note: a and b are distances from the center, c

The parameter e is known as the eccentricity. The value of this parameter defines the shape of our orbit. Depending on the value of e there are four kinds of shapes (conic sections), which means there are four kinds of orbits: circle, ellipse, parabola, and hyperbola, for e = 0, < 1, = 1, and > 1, respectively.

$H \equiv 1$ $\quad e_e := .6$ $\quad e_c := 0$ $\quad e_h := 2$ $\quad e_p := 1.000$ $\quad \theta := 0, 0.01..2\pi$ $\quad G\cdot m_e = 3.985\times10^{14}\frac{m^3}{s^2}$

Basics from Newton's Laws: Energy, Momentum, Parameters of Ellipse $\quad r_o := 300\text{km}$ $\quad \phi_o := 0$

$\text{Energy}(v,r) := \frac{v^2}{2} - \frac{\mu}{r}$ $\quad h(v_o, r_o, \phi_o) := r_o\cdot v_o\cdot\cos(\phi_o)$ $\quad h = r^2\cdot v\nu^2$ $\quad h_u(p) := \sqrt{\mu\cdot p}$

$p(v_o) := \frac{h(v_o, r_o, \phi_o)^2}{\mu}$ $\quad a(v_o) := \frac{-\mu}{\text{Energy}(v_o, r_o)}$ $\quad e_{traj}(v,a) := \sqrt{1 - \frac{p(v)}{a}}$

Period of Moon Sat Orbit

$T := 2\cdot\pi\cdot\sqrt{\frac{a_m{}^3}{G\cdot m_e}} = 99.98\cdot\text{hr}$

$r_h(\theta, e) := \frac{H}{1 + e\cdot\cos\left(\theta + \frac{\pi}{2}\right)}$

If we can Solve for Eccentric Anomaly, E, we get Time of Flight, TOF, t - T

$\cos(\nu) = \frac{p-r}{e\cdot r}$ $\quad \nu(p,r,e) := \text{acos}\left(\frac{p-r}{e\cdot r}\right)$

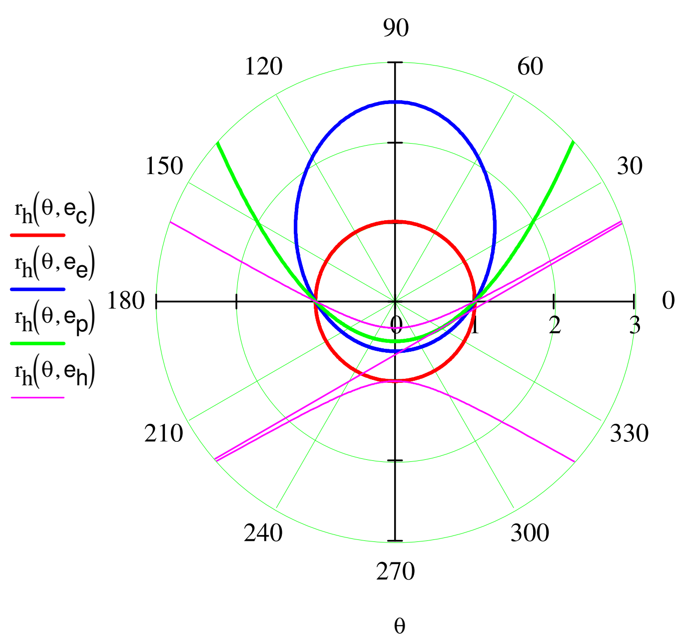
Recursion for Eccentric Anomaly, M & E (Deg)

mean anomaly M (in deg (0<= M<=360)

$MA(M_o, t, t_o) := M_o + \sqrt{\frac{\mu}{a_m{}^3}}\cdot(t - t_o)$

```
EcA(e,M,dp) := | mx_it ← 30
               | i ← 0
               | K ← π/180
               | del ← 10^(−dp)
               | m ← M/360
               | m ← 2·π(m − floor(m))
               | E ← m  if e < 0.8
               | E ← π  otherwise
               | F ← E − e·sin(m) − m
               | while |F| > del ∧ i < mx_it
               |     | E ← E − F/(1 − e·cos(E))
               |     | F ← E − e·sin(E) − m
               |     | i ← i + 1
               | E ← E/K
```

Plot of Conic Orbits: c, e, p, h



Find E and φ In Degrees

t/T: $\frac{27}{360} = 0.075$ $\quad EcA(e_m, 27, 5) = 28.501$ $\quad \phi(e, EcA) := 90 - \frac{180\cdot\text{atan2}\left(\sqrt{1-e^2}\cdot\sin(EcA), \cos(EcA) - e\right)}{\pi}$

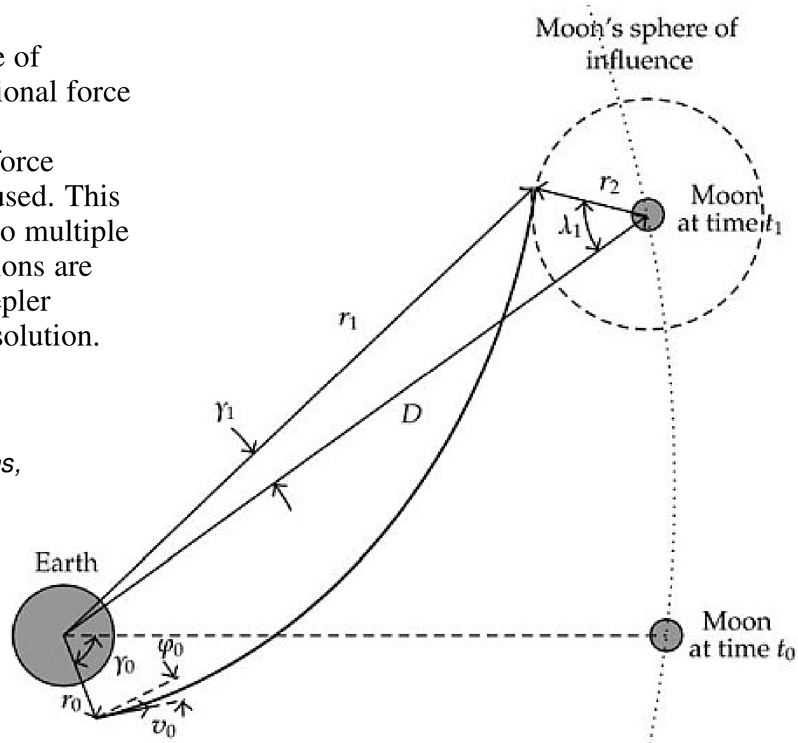
$\phi(0.977, 48.43418\cdot\text{deg}) = 153.029$

The Patched Conic Section Approximation for Finding a Lunar Trajectory

The Patched Conic Method is an Approximation for finding a trajectory by dividing space between the sphere of influence (SOI) of the earth, Lunar Earth Orbit (LEO) and the SOI region of the moon.

When the spacecraft is within the sphere of influence of the moon, only the gravitational force between the spacecraft and the moon is considered, otherwise the gravitational force between the spacecraft and the earth is used. This reduces a complicated n-body problem to multiple two-body problems, for which the solutions are the well-known conic sections of the Kepler orbits. Below is an example composite solution.

See for Example:
Optimal Two-Impulse Trajectories with Moderate Flight Time for Earth-Moon Missions,
Sandro da Silva Fernandes
Mathematical Problems in Engineering
Vol. 2012, Article ID 971983,
or
Bate, R. R., D. D. Mueller, and J. E. White, *Fundamentals of Astrodynamics*



Rather than dealing with large powers of 10, we can use **Astronomical Units**, for distance, velocity, time: AU, VU, TU. Where AU is the mean distance of the earth to the sun and DU is the radius of the earth. TU is the time unit. Then the velocity unit, (VU) is equal to DU/TU.

$DU := 6378.145km$ $\quad AU := 1.496 \cdot 10^8 km$ $\quad kmps := \frac{km}{s}$ $\quad VU := 7.905368kmps$ $\quad TU := 806.8s$ $\quad D := d_m$

Laplace's Equation for Moon's Sphere of Influence:
this is about 1/6 of the distance, D, to the moon

$R_{sif} := D \cdot \left(\frac{m_m}{m_e}\right)^{0.4}$ $\quad R_s := 66300km$ $\quad R_s = 10.395 \cdot DU$

The conic patched problem for finding a trajectory can be stated as follows:

Given: Initial rocket launch conditions in the earth's sphere of Influence, that is, initial position, velocity, flight path angle, and phase angle: $\mathbf{r_0}$, $\mathbf{v_0}$ $\mathbf{\phi_0}$, **and** $\mathbf{\gamma_0}$,
The three quntities $\mathbf{r_0}$, $\mathbf{v_0}$ $\mathbf{\phi_0}$ will give us initial energy and anglular momentum.

Find: Arrival conditions at moon's Sphere of Influence: $\mathbf{r_1}$, $\mathbf{v_1}$ $\mathbf{\phi_1}$, $\mathbf{\lambda_1}$. $\quad$ $\mathbf{r_0}$, $\mathbf{v_0}$ $\mathbf{\phi_0}$, **and** $\mathbf{\lambda_1}$

The problem with assigning these initial points is that they may not give a satisfactory solution to match the arrival conditions. Our strategy is to use the arrival ange λ_1 to the moon's SOI as one of the independent condition .

Given the 3 initial conditions and one arrival condition as our **independent variables:**
These will move us into the radius of the moon's sphere of influence. Some trial and error may still be required.

EXAMPLE: See Bate, R. R., D. D. Mueller, and J. E. White, *Fundamentals of Astrodynamics*

Solution: Select the **Apollo 11 Flight Conditions** for initial conditions: $\mathbf{r_0}$, $\mathbf{v_0}$, $\mathbf{\phi_0}$ **and** $\mathbf{\lambda_1}$.

Given: $r_0 := DU + 334km$ $v_0 := 10.6kmps$ $\phi_0 := 0deg$ A reasonable angle to arrive at moon $\lambda_1 := 30deg$

Find: $\mathbf{r_1}$, $\mathbf{v_1}$, $\mathbf{\phi_1}$, $\boldsymbol{\gamma_1}$ (the last symbol, γ, is the Greek letter gamma, the Arrival Phase Angle at the Moon)

Initial Energy and Angular Momentum are $Energy(v_0, r_0) = -0.011 \cdot VU^2$ $h_0 := h(v_0, r_0, \phi_0) = 1.441 \cdot \frac{DU^2}{TU}$

$D = 60.268 \cdot DU$ By the Law of Cosines: $r_1(\lambda_1) := \sqrt{D^2 + R_s^2 - 2D \cdot R_s \cdot \cos(\lambda_1)}$ $r1 := r_1(\lambda_1) = 51.529 \cdot DU$

From Law of Conservation of Energy and Momentum: $E_0 := Energy(v_0, r_0)$ $E_0 = -0.011 \cdot \frac{DU^2}{TU^2}$ $h_1 := h_0$

$v_1(r_1) := \sqrt{2 \cdot \left(E_0 + \frac{\mu}{r_1}\right)}$ $v1 := v_1(r1) = 0.128 \cdot VU$ $v1 := 0.1296VU$ $\phi_1 := acos\left(\frac{h_1}{r1 \cdot v1}\right)$ $\phi_1 = 77.542 \cdot deg$

In order to calculate the **Time of Flight**, TOF, to the moon's SOI, we need to Find:

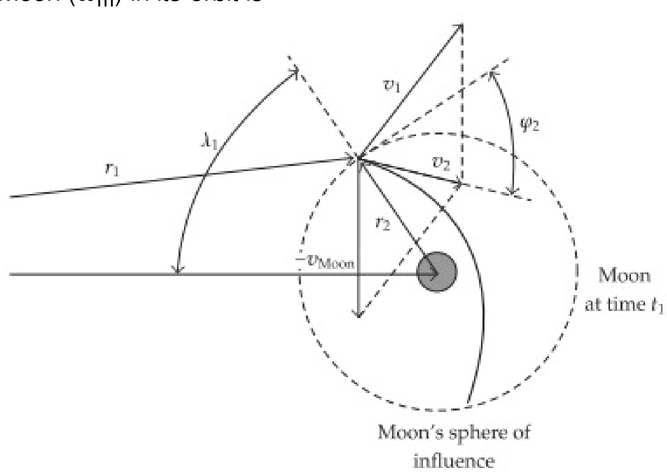
p, a, e, $\mathbf{E_0}$ **and** $\mathbf{E_1}$ for the Geocentric Trajectory.

$p := \frac{h_0^2}{\mu} = 2.075 \cdot DU$ $a := \frac{-\mu}{2\,Energy(v_0, r_0)}$ $e := \sqrt{1 - \frac{p}{a}}$ $e = 0.977$ $\nu_1 := \nu(p, r1, e)$ $\nu_1 = 2.956$

$\gamma_1 := asin\left(\frac{R_s}{r1} \sin(\lambda_1)\right) = 5.789 \cdot deg$ $a = 44.698 \cdot DU$ since: $\nu_0 := 0$ $EcA_0 := 0$ $EcA_1 := acos\left(\frac{e + \cos(\nu_1)}{1 + e \cdot \cos(\nu_1)}\right)$

$EcA_1 = 1.728$ $TOF := \sqrt{\frac{a^3}{\mu}} \cdot \left[\left(EcA_1 - e \cdot \sin(EcA_1)\right) - \left(EcA_0 - e \cdot \sin(EcA_0)\right)\right]$ $TOF = 51.132 \cdot hr$

We can use the same procedure at the moon (Selenocentric).
See Section XVI for the Newtonian Gravitational Solution for the Lunar Trajectory.
We need to determine the values of v1 and Rs in units based on the moon's gravitational attraction parameters.
The Angular Velocity of the Moon (ω_m) in its orbit is



: $\omega_m := 2.649 \cdot 10^{-6} \frac{rad}{s}$ $\omega_m := 2.137 \cdot 10^{-3} \frac{1}{TU}$ $\gamma_0 := \nu_1 - \nu_0 - \gamma_1 - \omega_m \cdot TOF$ $\gamma_0 = 135.637 \cdot deg$

$v_{1m} := 1.024kmps$ $\mu_m := 4093 \frac{km^3}{s^2}$ $v_m := 1.018kmps$ Then $v_{2m} := 1.198kmps$

$\varepsilon_2 := 5.68deg$ $e_m := 2.078$ $r_p := 4105km$ $h_p := 2367km$ $R_s = 10.395 \cdot DU$

$\mu_m := 4.903 \cdot 10^3 \cdot \frac{km^3}{s^2}$

Time of Flight

Develop an algorithm to Calculate Time of Flight

$$TOF_{alg}(v_0, r_0, \phi_0, \lambda_1) := \begin{array}{|l} h_0 \leftarrow r_0 \cdot v_0 \cdot \cos(\phi_0) \\ p \leftarrow \dfrac{h_0^{\,2}}{\mu} \\ E_0 \leftarrow \text{Energy}(v_0, r_0) \\ EcA_0 \leftarrow 0 \\ a \leftarrow \dfrac{-\mu}{2E_0} \\ e \leftarrow \sqrt{1 - \dfrac{p}{a}} \\ r1 \leftarrow \sqrt{D^2 + R_s^{\,2} - 2D \cdot R_s \cdot \cos(\lambda_1)} \\ \nu_1 \leftarrow \nu(p, r1, e) \\ EcA_1 \leftarrow \text{acos}\left(\dfrac{e + \cos(\nu_1)}{1 + e \cdot \cos(\nu_1)}\right) \\ TOF \leftarrow \dfrac{\sqrt{\dfrac{a^3}{\mu}} \cdot \left[\left(EcA_1 - e \cdot \sin(EcA_1)\right) - \left(EcA_0 - e \cdot \sin(EcA_0)\right)\right]}{hr} \\ A \leftarrow (TOF \;\; e)^T \end{array}$$

This gives a different value

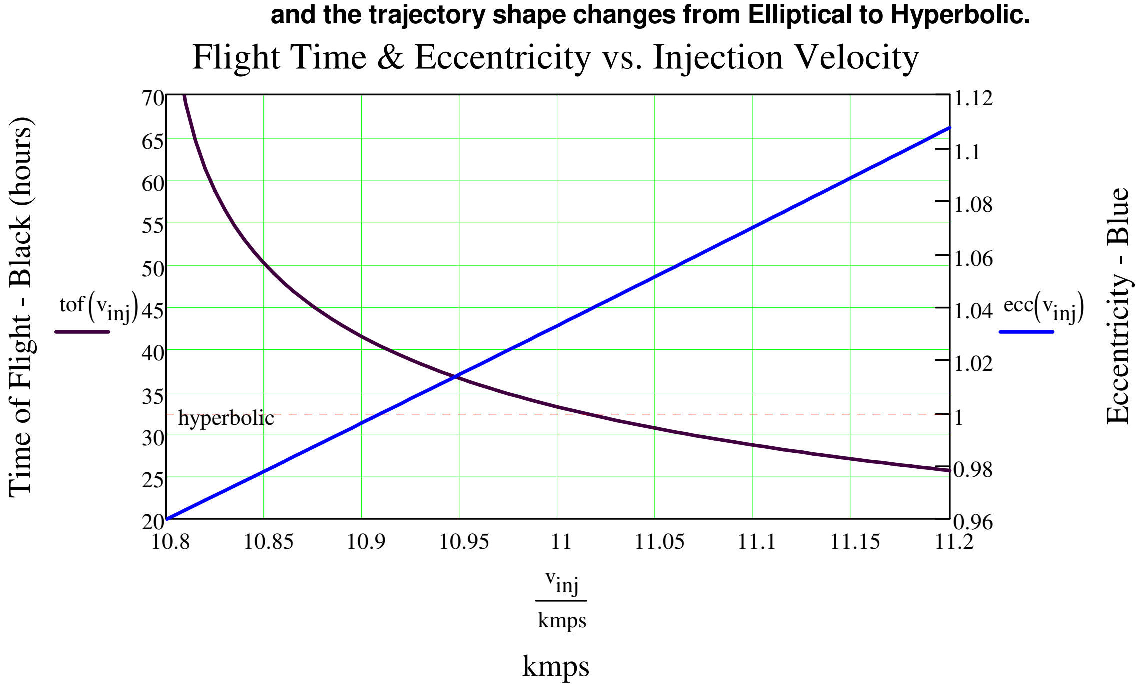
$v_0 = 10.846 \cdot kmps$ $\qquad TOF_{alg}(v_0, r_0, \phi_0, \lambda_1) = \begin{pmatrix} 51.132 \\ 0.977 \end{pmatrix}$

$tof(v_0) := TOF_{alg}(v_0, r_0, \phi_0, \lambda_1)_0 \qquad ecc(v_0) := TOF_{alg}(v_0, r_0, \phi_0, \lambda_1)_1$

Initial Conditions: $r_0 = 1.05 \cdot DU$ $\qquad$ $Alititude := r_0 - 1DU = 318.907 \cdot km$ $\qquad \phi_0 = 0$

$hyperbolic := 1$ $\qquad v_{inj} := 10.8kmps, 10.805kmps .. 11.2kmps$

Note: As the velocity increases above the minimum 10.8 kmps, the Time of Flight decreases and the trajectory shape changes from Elliptical to Hyperbolic.



Polar Plot of the Solution for the Patched Conic Lunar Approximation

$\nu := -90.002deg, -90.001deg .. 41deg$ $\chi := 39.5deg, 39.501deg .. 360deg$ $r(\nu) := \frac{a\cdot(1-e^2)}{1+e\cdot\cos(\nu+\gamma_0)}$

Note: λ_1 is not = 30 deg $\varphi := 33deg$ $Earth(\theta) := 1.5\sin(\theta+\varphi)$ $\theta := 0, 0.001 .. 2\pi$

$r_m := 82$ $a_m := 1.5$ $r_{moon}(\theta,\varphi) := r_m\cdot\cos(\theta-\varphi) + \sqrt{a_m^{\,2} - r_m^{\,2}\sin(\theta-\varphi)^2}$

Radius of Moon Sphere of Influence $r_{msi}(\theta,\varphi) := r_m\cdot\cos(\theta-\varphi) + \sqrt{10.4^2 - r_m^{\,2}\sin(\theta-\varphi)^2}$

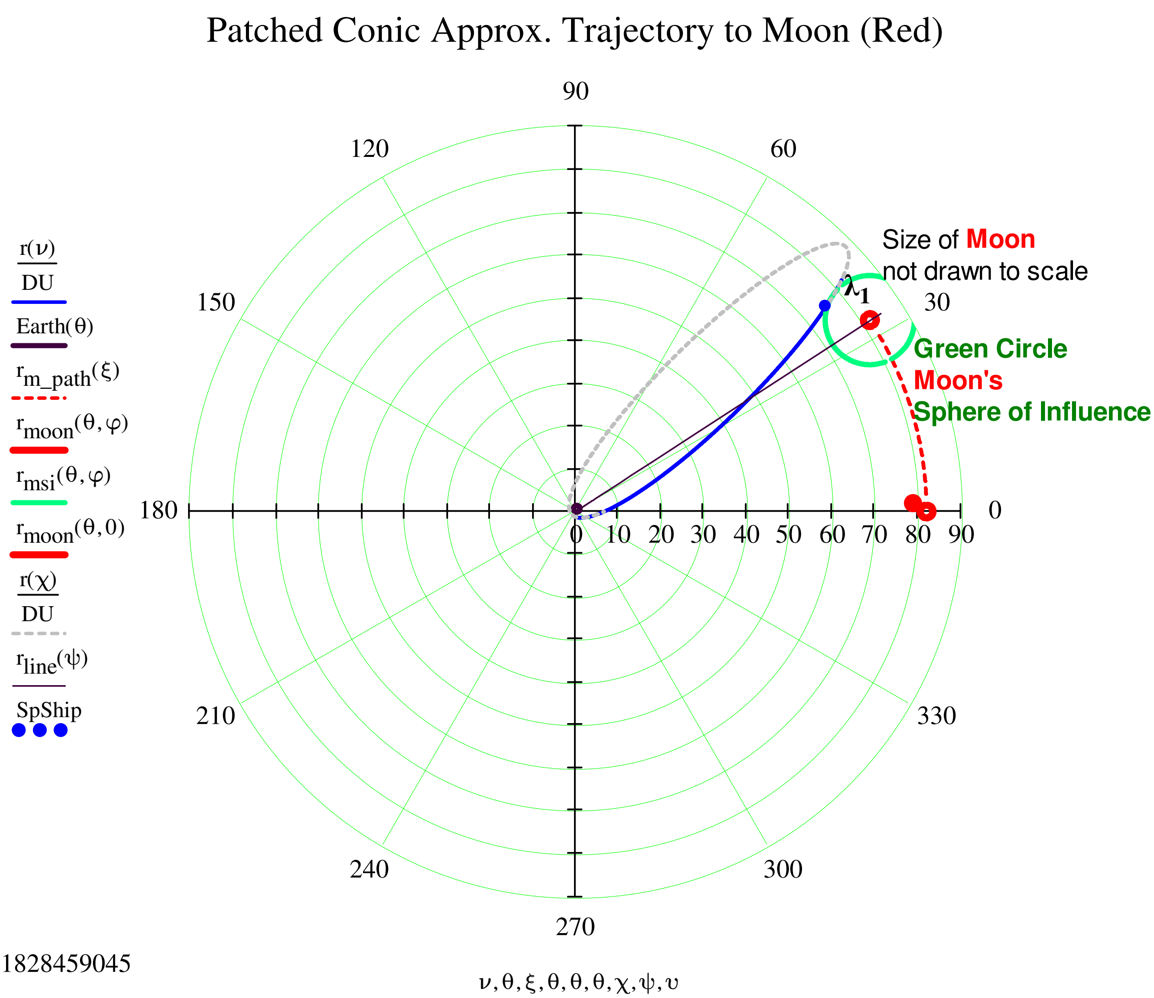
$\xi := 0.05, 0.051 .. \varphi - 0.05$ $r_{m\_path}(\xi) := r_m$

Point of Conic Patch
$SpShip := 75.5$

$\psi := 0, 0.0017365 .. \varphi$ $r_{line}(\theta) := \frac{0.1}{\sqrt{1-(1\cdot\cos(\theta-\varphi))^2}}$ $\upsilon := 39.5deg$

Polar Plot: Geocentric Frame - Earth at the Center

From the list of functions shown on the left of the plot below:
r(v) shows the Trajectory Ellipse Conic Ptach in blue, Earth(θ) is at the center in black, $r_{moon}(\theta,\varphi)$ in red is the location of the moon at intercept $\varphi = 33^o$, $r_{msi}(\theta)$ is the circle in green of the moon's of sphere of influence, $r_{moon}(\theta,0)$ in red is the initial location of the moon at 0^o, $r_{m\_path}(\xi)$ is the dotted line path of moon from 0 to φ. r(χ) is the dotted line that shows the elliptical path back to the earth, and r_{line} is the red straight line from earth at center to the moon to show angle λ_1. SpCraft is where SpaceCraft enters the Moon's Sphere of Influence. Point of Conic Patch. Blue dot.



$e := 2.718281828459045$

IA. Apollo Free Return Trajectory: Simulation for CSM to Moon & Back

Trajectory Model: 3-Body (Earth, Moon, Spacecraft) 2D Planar Point Mass with **Earth at Center**

This 3 body gravitational solution for the FRT uses the Mathcad Differential Equation Solving Methodology discussed: arXiv:1504.07964
"***Motion of the planets: the calculation and visualization in Mathcad***", Valery Ochkov, Katarina Pisa

The aborted Apollo 13 mission was the only mission to actually turn around the Moon in a free-return trajectory.

Solve the Gravitational and Dynamics Equations for Earth, Moon, & CSM Trajecto

$kg := 1$ $m := 1$ $s := 1$ $N := 1$ $s := 1$ $min := 60s$ $hr := 3600s$ $kgf := 9.80665N$

$km := 1000m$ $kmps := km$ $kph := \frac{km}{hr}$ $mph := 0.447 \cdot 10^{-3} kmps$ $G := 6.67384 \cdot 10^{-11} \cdot \frac{N \cdot m^2}{kg}$

Run Simulation for 160 hrs **Apollo 11 Orbit 77 hrs**

$FRAME := 999$ $n_{ode} := 20000$ $n := 999$ $n_{plot} := 10000$ $t_{end} := \frac{160hr}{n+1} \cdot (FRAME + 1)$ $t_{orb} = 81.44\ hr$

Time of Flight (TOF) = t_{orb}

Trajectory to Moon's Sphere of Influence

Initial x,y Velocity CSM **Radius of Earth** **Apogee to Moon**

$v_{0x} := 6.811kmps$ $v_{0y} := 6.356kmps$ $v_{CSM} := 9.317kmps$ $R_e := 6370km$ $d_{m\_ap} := 405500km$

Define Gravitational and Dynamics Equations for Earth, Moon, and CSM

Mass Start position Start Velocity

$$\begin{matrix} \text{Earth, e} \\ \text{Moon, m} \\ \text{CSM, s} \end{matrix} \begin{pmatrix} m_e & x_{e0} & y_{e0} & vx_{e0} & vy_{e0} \\ m_m & x_{m0} & y_{m0} & vx_{m0} & vy_{m0} \\ m_s & x_{s0} & y_{s0} & vx_{s0} & vy_{s0} \end{pmatrix} := \begin{pmatrix} 5.972 \cdot 10^{24} kg & 0m & 0m & 0kph & 0kph \\ 7.347 \cdot 10^{22} kg & d_{m\_ap} & 0km & 0kmps & 0.97kmps \\ 13600kg & R_e + 100km & R_e - 100km & v_{0x} & v_{0y} \end{pmatrix}$$

Given **Solve Set of Differential Guidance Equations for 3 Body Problem of Earth, Moon, and CSM**

$x_e(0) = x_{e0}$ $x_e'(0) = vx_{e0}$ $y_e(0) = y_{e0}$ $y_e'(0) = vy_{e0}$

$$m_e \cdot x_e''(t) = \frac{G \cdot m_e \cdot m_m \cdot (x_m(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_m(t))^2 + (y_e(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (x_s(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3}$$

$$m_e \cdot y_e''(t) = \frac{G \cdot m_e \cdot m_m \cdot (y_m(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_m(t))^2 + (y_e(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (y_s(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3}$$

$x_m(0) = x_{m0}$ $x_m'(0) = vx_{m0}$ $y_m(0) = y_{m0}$ $y_m'(0) = vy_{m0}$

$$m_m \cdot x_m''(t) = \frac{G \cdot m_m \cdot m_e \cdot (x_e(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_m \cdot m_s \cdot (x_s(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3}$$

$$m_m \cdot y_m''(t) = \frac{G \cdot m_m \cdot m_e \cdot (y_e(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]^3} + \frac{G\, m_m \cdot m_s \cdot (y_s(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3}$$

$x_s(0) = x_{s0}$ $x_s'(0) = vx_{s0}$ $y_s(0) = y_{s0}$ $y_s'(0) = vy_{s0}$

$$m_s \cdot x_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (x_e(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_e(t))^2 + (y_s(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (x_m(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_m(t))^2 + (y_s(t) - y_m(t))^2}\right]^3}$$

$$m_s \cdot y_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (y_e(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_e(t))^2 + (y_s(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (y_m(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_m(t))^2 + (y_s(t) - y_m(t))^2}\right]^3}$$

IA. Free Return Trajectory: 3 Body Sim for CSM to the Moon & Back

Trajectory Model: 3-Body (Earth, Moon, Spacecraft) 2D Planar Point Mass with **Earth at Center**

Differential Equation Solver

$$\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \end{pmatrix} := \text{Odesolve}\left[\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \end{pmatrix}, t, t_{end}, n_{ode}\right]$$

Initial Velocity (km/s) of CSM at an Altitude of 141 km:

$$\sqrt{v_{0x}^2 + v_{0y}^2} = 9.316 \cdot \text{kmps}$$

Distance to the Center of the Moon

$$d_m(t) := \left[\sqrt{\left(x_m(t) - x_s(t)\right)^2 + \left(y_m(t) - y_s(t)\right)^2}\right] \qquad \frac{d_m(t_{orb})}{R_m} = 1.619$$

time t_{fly}, just beyond lunar fly by time **t_{flydot}**

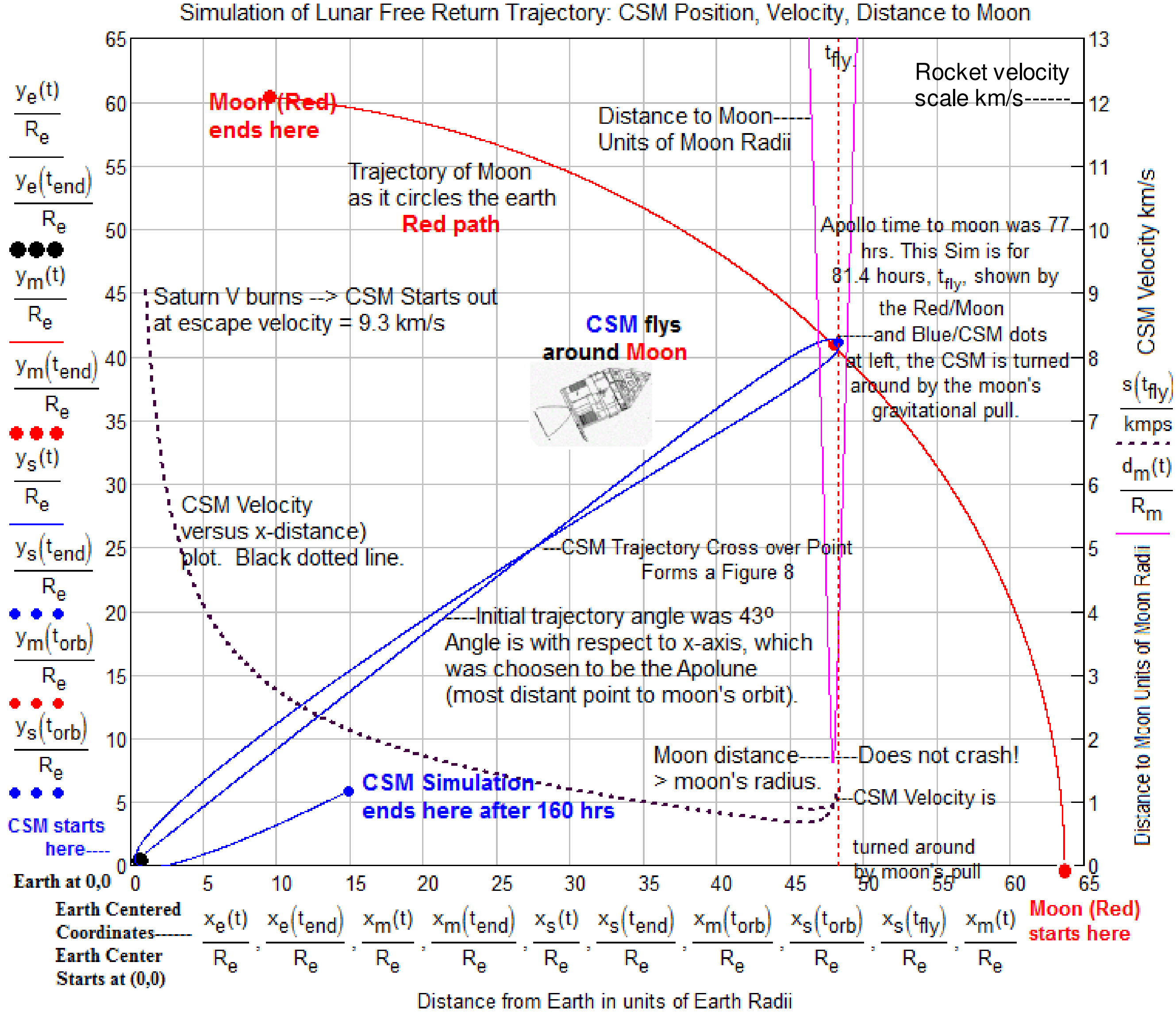
$$t := 0, \frac{t_{end}}{n_{plot}} .. t_{end} \qquad t_{fly} := 0, \frac{t_{end}}{n_{plot}} .. 90hr \qquad t_{fly.} := \frac{x_s(t_{orb})}{R_e}$$

$$vx_s(t) := \frac{d}{dt}x_s(t) \qquad vy_s(t) := \frac{d}{dt}y_s(t) \qquad s(t) := \sqrt{vx_s(t)^2 + vy_s(t)^2}$$

Finding a Free Return Trajectory (FRT) is a little tricky. First, the trajectory must catch the moon at the exact place and time as travels around the earth and then after being swing around by the moon's gravity it must swing back and catch the earth in such a way as to go into earth orbit. This can present a problem for the Differential Equation Solver. This is a three body problem. A change in the CSM's trajectory is influenced by the pull the moon, which in turn is affected by the pull of the earth. The solver can easily fail to converge on a solution. A change in angle by 10 degrees can result in a large change in orbit time of 4.5 days. We also must check that CSM does not crash into moon.

Below is a plot of our FRT solution for the Apollo Trajectory. It shows the CSM's x,y position and velocity from earth to moon and back. .Note the figure 8 orbit of this Free Return. The Apollo 11 flight time to the moon was 77 hours. Our simulation is for 81.4 hours. Because of instabilities, convergence problems, etc. some trial and error was required.

Simulation of Lunar Free Return Trajectory: CSM Position, Velocity, Distance to Moon



IB. 4-Body Sim of Apollo Free Return Trajectory: CSM to Moon & Back

Trajectory Model: 4-Body (Earth, Moon, Sun, Spacecraft) 2D Planar Point Mass w **Earth at Center**

This Simulation Uses the Mathcad Differential Equation Solving Methodology discussed in: arXiv:1504.07964
"***Motion of the planets: the calculation and visualization in Mathcad***", Valery Ochkov, Katarina Pisačić

4-Body Reference Frame: Earth and moon are initially at 0,0 and the earth and sun are initially not moving.

$kg := 1$ $m := 1$ $s := 1$ $N := 1$ $s := 1$ $min := 60s$ $hr := 3600s$ $kgf := 9.80665N$

$km := 1000m$ $kmps := km$ $kph := \frac{km}{hr}$ $mph := 0.447 \cdot 10^{-3} kmps$ $n_{plot} := 10000$

Run Simulation for 115 hrs **Apollo 11 Orbit 77 hr**

$FRAME := 999$ $n_{ode} := 20000$ $n := 999$ $t_{end} := \frac{114.5hr}{n+1} \cdot (FRAME + 1)$ $t_{orb} := 58.5hr$

Time of Flight (TOF) = t_{orb}

$G := 6.67384 \cdot 10^{-11} \frac{N \cdot m^2}{kg}$

Trajectory to Moon's Sphere of Influence Apolune

Initial x,y Velocity CSM **Radius of Earth** **Apogee to Moon**

$v_{0x} := 7.58kmps$ $v_{0y} := 5.5kmps$ $R_m := 1737.4km$ $R_e := 6370km$ $d_{m\_ap} := 405500km$

$t_{end} = 114.5hr$ $v_{CSM} := \sqrt{v_{0x}^2 + v_{0y}^2}$ $v_{CSM} = 9.365 \cdot kmps$ $d_{e\_ap} := 152 \cdot 10^6 km$

Define Gravitational and Dynamics Equations for Earth, Moon, and CSM

e is Earth
a is Sun
m is Moon
s is CSM

$$\begin{pmatrix} m_e & x_{e0} & y_{e0} & vx_{e0} & vy_{e0} \\ m_a & x_{a0} & y_{a0} & vx_{a0} & vy_{a0} \\ m_m & x_{m0} & y_{m0} & vx_{m0} & vy_{m0} \\ m_s & x_{s0} & y_{s0} & vx_{s0} & vy_{s0} \end{pmatrix} := \begin{pmatrix} 5.972 \cdot 10^{24} kg & 0m & 0m & 0kph & 0kmps \\ 1.989 \cdot 10^{30} kg & -130 \cdot 10^6 km & -80 \cdot 10^6 km & 0kmps & 0kmps \\ 7.347 \cdot 10^{22} kg & d_{m\_ap} & 0km & 0kmps & 0.97kmps \\ 13600kg & R_e + 110km & R_e - 96km & v_{0x} & v_{0y} \end{pmatrix}$$

Given

Set of Differential Guidance Equations for 4 Body Problem of Earth, Moon, and CSM

$x_e(0) = x_{e0}$ $x_e'(0) = vx_{e0}$ $y_e(0) = y_{e0}$ $y_e'(0) = vy_{e0}$ $x_m(0) = x_{m0}$ $x_m'(0) = vx_{m0}$ $y_m(0) = y_{m0}$ $y_m'(0) = vy_{m0}$

$$m_e \cdot x_e''(t) = \frac{G \cdot m_e \cdot m_m \cdot (x_m(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_m(t))^2 + (y_e(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (x_s(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (x_s(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3}$$

$$m_e \cdot y_e''(t) = \frac{G \cdot m_e \cdot m_m \cdot (y_m(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_m(t))^2 + (y_e(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (y_s(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (y_s(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3}$$

$$m_m \cdot x_m''(t) = \frac{G \cdot m_m \cdot m_e \cdot (x_e(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_m \cdot m_s \cdot (x_s(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_m \cdot m_s \cdot (x_s(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]}$$

$$m_m \cdot y_m''(t) = \frac{G \cdot m_m \cdot m_e \cdot (y_e(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]^3} + \frac{G m_m \cdot m_s \cdot (y_s(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3} + \frac{G m_m \cdot m_s \cdot (y_s(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]}$$

$x_s(0) = x_{s0}$ $x_s'(0) = vx_{s0}$ $y_s(0) = y_{s0}$ $y_s'(0) = vy_{s0}$ $x_s(0) = x_{s0}$ $x_s'(0) = vx_{s0}$ $y_s(0) = y_{s0}$ $y_s'(0) = vy_{s0}$

$$m_s \cdot x_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (x_e(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_e(t))^2 + (y_s(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (x_m(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_m(t))^2 + (y_s(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_s \cdot (x_s(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_s(t))^2 + (y_s(t) - y_s(t))^2}\right]^3}$$

$$m_s \cdot y_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (y_e(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_e(t))^2 + (y_s(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (y_m(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_m(t))^2 + (y_s(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_s \cdot (y_s(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_s(t))^2 + (y_s(t) - y_s(t))^2}\right]^3}$$

$$m_s \cdot x_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (x_e(t) - x_s(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (x_m(t) - x_s(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_s \cdot (x_s(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_s(t))^2 + (y_s(t) - y_s(t))^2}\right]^3}$$

$$m_s \cdot y_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (y_e(t) - y_s(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (y_m(t) - y_s(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_s \cdot (y_s(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_s(t))^2 + (y_s(t) - y_s(t))^2}\right]^3}$$

IB. 4-Body Sim of Apollo Free Return Trajectory: CSM to Moon and Back

Trajectory Model: 4-Body (Earth, Moon, Sun, Spacecraft) 2D Planar Point Mass w **Earth at Center**

Plot for Sim of 4-Body Free Return Traj: CSM to Moon and Back

Differential Equation Solver

Earth, Moon, Space Craft, Sun

$$\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \\ x_a \\ y_a \end{pmatrix} := \text{Odesolve}\left[\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \\ x_a \\ y_a \end{pmatrix}, t, t_{end}, n_{ode}\right]$$

Initial Velocity (km/s) of CSM at an Altitude of 141 km:

$$\sqrt{v_{0x}^2 + v_{0y}^2} = 9.365 \cdot \text{kmps}$$

time t_{fly}, just beyond lunar fly by time **t flyby dot**

$$t := 0, \frac{t_{end}}{n_{plot}} .. t_{end} \qquad t_{fly} := 0, \frac{t_{end}}{n_{plot}} .. 160\text{hr} \qquad t_{fly.} := \frac{x_s(t_{orb})}{R_e}$$

$$vx_s(t) := \frac{d}{dt} x_s(t) \qquad vy_s(t) := \frac{d}{dt} y_s(t) \qquad s(t) := \sqrt{vx_s(t)^2 + vy_s(t)^2}$$

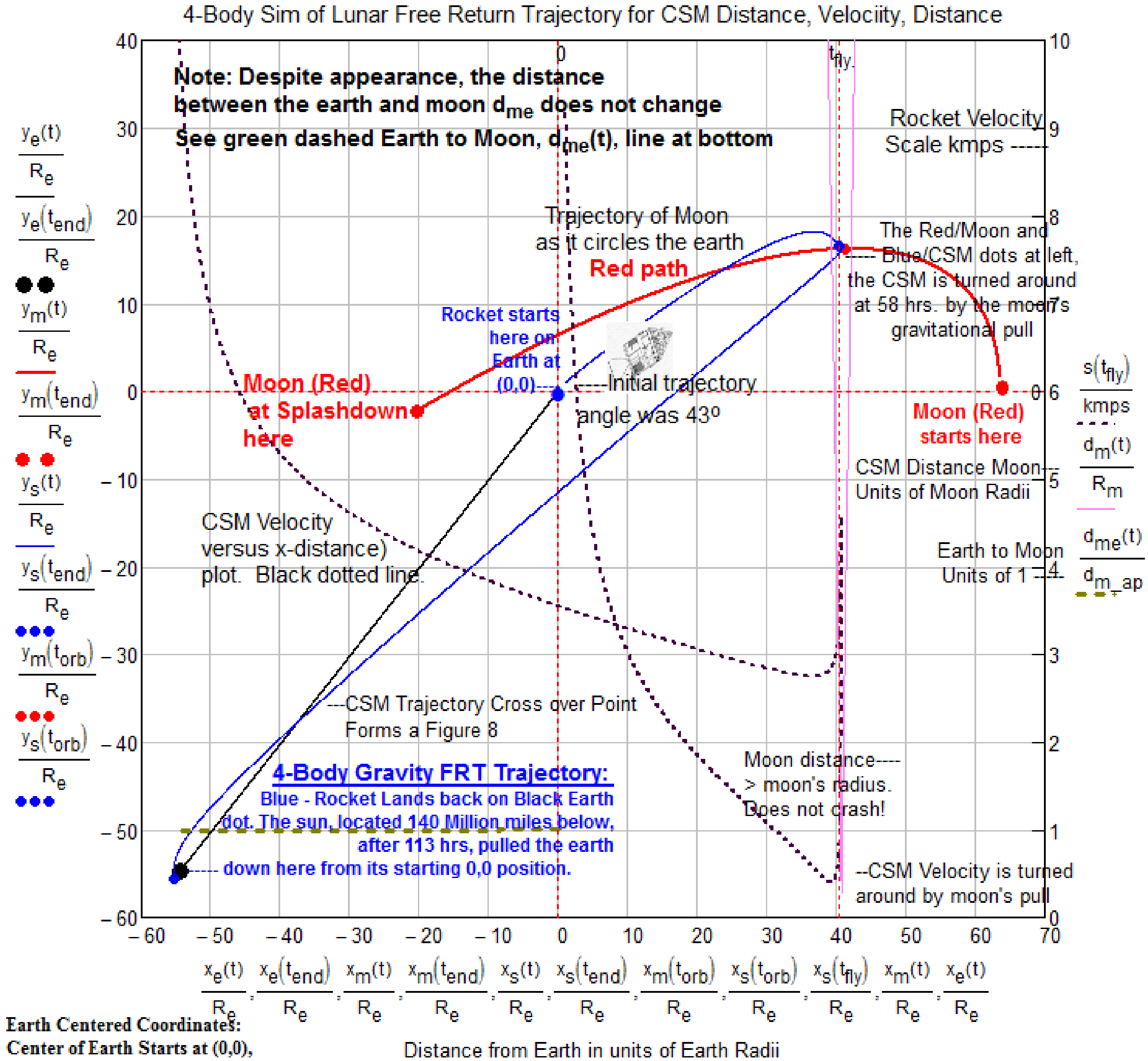
Distance from Earth to Moon

$$d_{me}(t) := \left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]$$

Distance to the Center of the Moon

$$d_m(t) := \left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]$$

$$\frac{d_m(t_{orb})}{R_m} = 3.242$$



Earth Centered Coordinates: Center of Earth Starts at (0,0), but gravitational pull of sun, 94 million miles below-left of earth pulls the earth down & left from 0,0 so it ends at black dot above. The rocket (blue dot)lands back on earth 114 hours after launch.

Note: The radial velocity of the earth around the sun is 1º every 365 days or 1/365º per day. Our sim runs 114 hrs or 114/24 days. This results in (1/365º) x 114/24 or 7.5º. For the purpose of our illustration, we will ignore this added complexity. Think of this as a rotating reference frame, such as our experience of us living on a rotating earth