

III. Mathematical Basis Λ CDM Cosmology: Einstein's General Relativity, FLRW & GR Tests

In 1915, Einstein developed his **General Theory of Relativity (GR)**. GR consists of a number of field equations that relate the geometry of spacetime to the distribution of matter within it. GR provides a deep physical and geometrical description of how mass/energy determines the dynamics of the universe.

The space-time evolution of the universe is guided by the Einstein Field Equation.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{M_{\text{pl}}^2}T_{\mu\nu}$$

where the spacetime metric $g_{\mu\nu}$ and its corresponding Ricci tensor $R_{\mu\nu}$ and Ricci scalar R are related to energy content expressed through the Einstein Symmetric, order-2, Energy-Momentum Tensor $T_{\mu\nu}$. Briefly, the Einstein equations equate the matter that's present in a spacetime with the spacetime's geometry.

In 1917, Schwarzschild solved the Einstein equations under the assumption of spherical symmetry, two years after their publication. The most obvious spherically symmetric problem is that of empty space outside a planet or star. The mass curves space-time and thus affects the particles moving nearby. The space-time interval in spherical coordinates in the Schwarzschild solution is.

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} - r^2 d\theta^2 - r^2 \sin^2 \theta d\psi^2$$

General Relativity Test #1: Schwarzschild Equation Prediction of the Formation of a Black Hole

The definition of **proper time**, τ (tau), is the time interval for an **observer at rest**. In Minkowski space time

$ds^2 = dt^2 - dx^2$, for $dx = 0$, $dt = d\tau$. Similarly in the Schwarzschild Metric if we have an observer at rest then

$dr = 0$...etc and then proper time should be $d\tau = dt$ (like in the SR case giving $ds = (1-2M/r)dt = (1-2M/r)d\tau$

the first term T_{00} in the Einstein $T_{\mu\nu}$ Tensor is T_{tt} . If you take the distance r to be equal to $r_s = 2 \frac{G \cdot M}{c^2}$

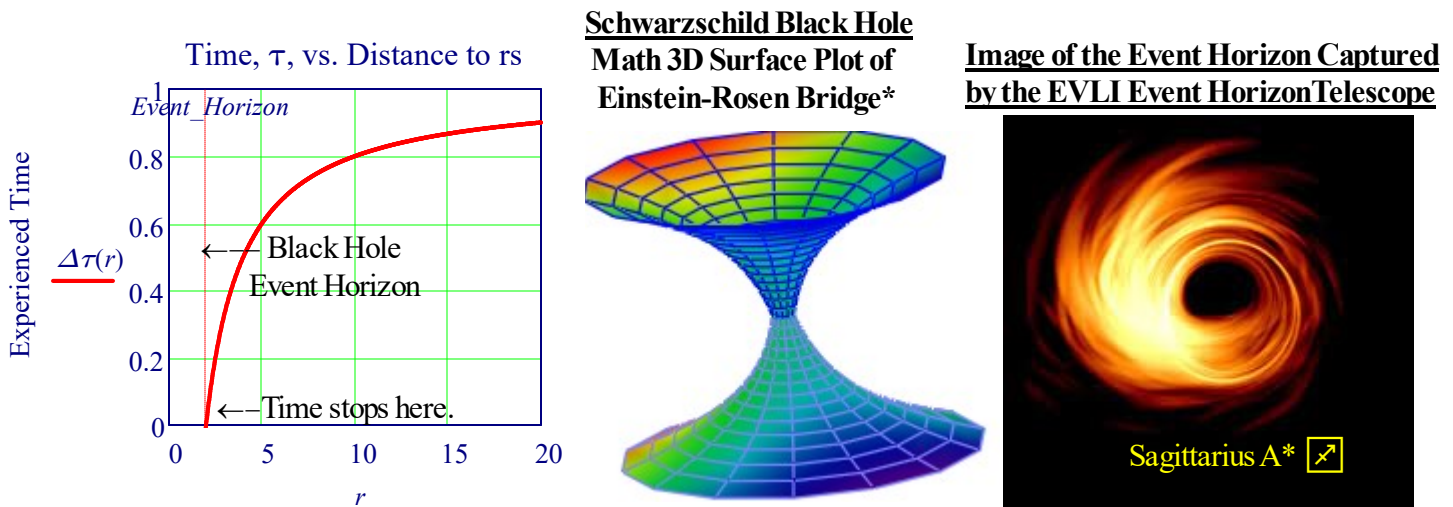
then the time factor, T_{tt} , which is equal to $1 - 2GM/c^2 r$ in ds^2 becomes 0, so the value of ds^2 is undefined.

It becomes a singularity. This value of the **radius = r_s** is called the **Schwarzschild radius**.

From the Schwarzschild Metric, if we plot the passage of time, $\Delta\tau$, versus the distance to the center, r , the relation is:

$$\Delta\tau(r) := 1 - 2 \frac{GM}{r} \quad E-R(\chi) := \chi \cdot e^{\frac{\chi}{2}} + \Theta \quad \text{See Plots Below for Passage of Time } \Delta\tau(r) \text{ and Space (ER) in Schwarzschild Black Hole}$$

The Plot below show that at the Event Horizon that the passage of proper time, τ , slows to 0, that is, time stops.



Theoretically, this connects two asymptotically flat "universes."

* Physikalische Zeitschrift. XVII: 448; Einstein & Rosen 1935, Phys. Rev. 48 73

General Relativity Test #2: GR Calculation of Precession of Mercury's Orbit

Reference: *The Precession of Mercury's Perihelion*, Owen Biesel, <https://sites.math.washington.edu>

Our First Test is the Calculation of the Precession of the Perihelion of Mercury. Newton's Theory says it 532 arcseconds per century, but the observed value is 43 arcseconds per century.

We will apply a General Relativistic treatment of geodesics in the Schwarzschild metric, and show that an "orbit" matches the observed Mercury's shift of approximately **43 arcseconds per century**. We assume that the particle is a test particle traveling along a geodesic through spacetime. Geodesics can also be described as stationary points of the integral

$$I = \int \langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle d\tau$$

Assume that the metric that similar to the above Schwarzschild Solution, we assume that the solar system is spherically symmetric, static, and asymptotically flat, so that the metric can be represented as follows:

$$ds^2 = -e^{2\alpha(R)} dT^2 + e^{2\beta(R)} dR^2 + e^{2\gamma(R)} d\Omega^2,$$

where the $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ term comes from spherical symmetry and T is the coordinate produced by the timelike Killing vector field. Then the **Euler-Lagrange Equations** for ϕ and T are

Binding Energy per Unit Mass, -E

$$-E = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\phi}^2) - \frac{GM}{r}$$

$$0 = \frac{d}{d\tau} (2r^2 \dot{\phi})$$

$$0 = \frac{d}{d\tau} \left(-2 \left(1 - \frac{R_S}{r} \right) \dot{T} \right)$$

Now $r(\phi)$ is periodic with period 2π .
where $R_S = 2GM$ is the Schwarzschild radius of the sun.

The above implies that $L = r^2 \dot{\phi}$ and $E = T'(R_S/r - 1)$ are two constants of the motion. Then the relation $\mathcal{L} = -1$ gives us:

$$(\dot{r})^2 = (E^2 - 1) + \frac{R_S}{r} - \frac{L^2}{r^2} + \frac{R_S L^2}{r^3}$$

Now we introduced the notation R_{\pm} for the nonzero roots of the quartic polynomial in terms of the closed orbit r'

Now the requirement that of a closed orbit with $(r')^2 \geq 0$ imposes some constraints on L , E , and R_S ; we need a connected component of $\{r : r' \geq 0\}$ to be a compact subset of R_+ .

This means there exist at least two values R_+ and R_- where $r' = 0$, i.e. aphelion and perihelion. For Mercury

$$M_{\odot} := 1.989 \cdot 10^{30} \text{ kg} \quad R_+ := 69.8 \cdot 10^6 \text{ km} \quad R_- := 46 \cdot 10^6 \text{ km} \quad R_S := 2.95 \text{ km}$$

Then the angle shift from R_- to R_+ is given, as in the Classical case, by:

$$\phi_+ - \phi_- = \int_{R_-}^{R_+} \frac{dr}{\sqrt{\frac{E^2-1}{L^2} r^4 + \frac{R_S}{L^2} r^3 - r^2 + R_S r}}$$

Define D, ε :

Now use the Taylor series expansion $(1 - \varepsilon/r)^{-1/2} \approx 1 + \varepsilon/2r$. This gives $E := 0$

$$D := \frac{R_+ \cdot R_-}{R_+ + R_-}$$

$$\phi_{\pm} = \phi_+ - \phi_- = \sqrt{\frac{L^2}{1-E^2}} \int_{R_-}^{R_+} \frac{1 + \varepsilon}{r \sqrt{(R_+ - r)(r - R_-)}} + \frac{\varepsilon/2}{r^2 \sqrt{(R_+ - r)(r - R_-)}} dr$$

$$\varepsilon := \frac{R_S}{1 + R_S \cdot D^{-1}}$$

$$\phi_{\pm} := \frac{\pi}{\sqrt{1 - \frac{R_S}{D}}} \cdot \left(1 + \frac{1}{4} \frac{R_S \cdot D^{-1}}{1 - R_S \cdot D^{-1}} \right) + \frac{\pi}{\sqrt{1 - \frac{R_S}{D}}} \cdot E$$

$$\phi_{\pm} - \pi = 0$$

The above ϕ_{\pm} equation is a trustworthy estimate of $\phi_+ - \phi_-$ (half a revolution, in radians)

Since Mercury completes 415.2 revolutions each century, and there are $360 \cdot 60 \cdot 60 / 2\pi$ arcseconds per radian, we find that General Relativity predicts that Mercury's perihelion advances by

$$(\phi_{\pm} - \pi) \cdot \frac{360 \cdot 60 \cdot 60}{\pi} \cdot 415.2 = 42.938 \text{ arcseconds per century.}$$

This calculated value from General Relativity agrees with the observed value of 43.1 ± 0.5 arcseconds per century.

GR Test #3: Predict Clock Difference Between GPS Satellite & Surface of Earth

The difference between a clock on the surface of earth and a clock in a Global Position Satellite in orbit above the earth is 38 microseconds per day. Does General Relativity predict this value of 38 microsecond difference?

Two Things Affect the Net Time Dilation (Note: This also applies to distant starlight. See Section XXIB):

Gravitation and Velocity. Velocity of Satellite slows satellite time down relative to earth, but the Earth's Gravitational Field Slows down clocks on earth for different heights.

G is Universal Gravitational Constant

$$G := 6.674 \cdot 10^{-11} \cdot m^3 \cdot kg^{-1} \cdot s^{-2}$$

$$v_{eq} := 0.465 \frac{km}{s} = 1040.175 \cdot \frac{mile}{hr}$$

GPS Period is 12 hours

$$v_s := 3697 \frac{m}{s} = 8269.953 \cdot \frac{mile}{hr}$$

M is mass of the earth

$$g = 9.807 \frac{m}{s^2}$$

$$r_{eq} := 6378 km$$

$$alt_s := 20200 km$$

$$alt_s = 12551.698 \cdot mile$$

c is the speed of light

$$c = 2.99792458 \times 10^8 \frac{m}{s}$$

$$M_e := 5.97 \cdot 10^{24} \cdot kg$$

Scale Factor, Micro, μ

$$\mu := 10^{-6}$$

Special and General Relativity Gives the Amount of Time Dilation as:

General Relativity: Schwarzschild Metric Gravitational Time Dilation Per Day

The Einstein Field Equation

Gives the Schwarzschild Metric

$$R_{\mu, \nu} - \frac{g_{\mu, \nu} R}{2} - \lambda g_{\mu, \nu} = \kappa T_{\mu, \nu}$$

The Schwarzschild Metric, describes space-time in the vicinity of a non-rotating massive spherically symmetric.

It gives the change in time, $\Delta\tau_{\Delta t_{gravity}}$ for altitude, alt.

$$\Delta\tau_{\Delta t_{gravity}}(alt) := \left[\sqrt{1 - \frac{2 \cdot G \cdot M_e}{(alt + r_{eq}) \cdot c^2}} - \sqrt{1 - \frac{2 \cdot G \cdot M_e}{r_{eq} \cdot c^2}} \right] \cdot 24 \cdot 60 \cdot 60s$$

$$\Delta\tau \text{ for Satellite at Altitude: } \Delta\tau_{\Delta t_{gravity}}(alt_s) = 45.643 s \cdot \mu$$

Special Relativity: Velocity Time Dilation Per Day

$$v_{orbit}(r) := \sqrt{\frac{G \cdot M_e}{r}}$$

$$\Delta\tau_{\Delta t_{velocity}}(r) := 1 - \frac{1}{\sqrt{\left(1 - \frac{v_{orbit}(r)^2}{c^2}\right)}}$$

$$\Delta t_{velocity_day} := \Delta\tau_{\Delta t_{velocity}}(alt_s + r_{eq}) \cdot 24 \cdot 60 \cdot 60s = -7.206 s \cdot \mu$$

Total Time on Earth Per Day is (Dilated) Longer by microseconds:

$$Total \text{ GPS Clock Dilation} := \Delta\tau_{\Delta t_{gravity}}(alt_s) + \Delta t_{velocity_day}$$

$$Total \text{ GPS Clock Dilation} = 38.438 s \cdot \mu$$

Twice a Day the Time for a Satellite is Slowed by (19 μ s) to match time on Earth
If not Corrected the Position Error Per Day Would be:

Two Satellite GPS Distance Error = Time Error x Speed of Light, c

$$Total \text{ GPS Clock Dilation} \cdot c = 7.16 \cdot mile$$