IV. The Equation of State for a Simple Fluid Model

• Usually written as P = w ρ P is the Pressure and ρ is the density.
• Note that this relationship is the simplest model. The actual model may be more complex.
• This is not necessarily the best way to describe matter/energy density; it implies a fluid of some kind
This may be acceptable for the matter and radiation we know,
but maybe it is not an optimal description for the dark energy
• Define Special values:
w = 0 means P = 0, e.g., non-relativistic matter
w = 1/3 is radiation or relativistic matter
w = −1 looks just like a cosmological constant
… but it can have in principle any value, and it can be changing in redshift

Evolution of the Density, ρ

Generally: $\rho \approx a^{-3(w+1)}$

Continuity Equation
(Specifies that matter is conserved.)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \boldsymbol{v}) = 0$$

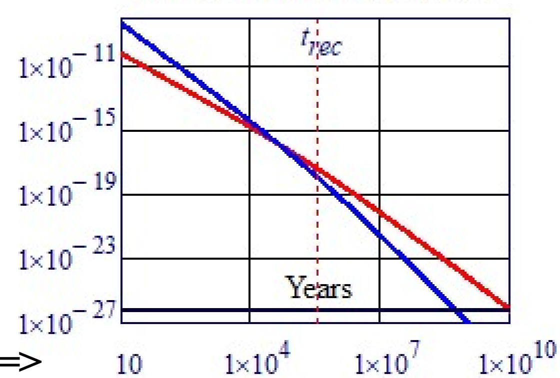
• Matter dominated (w = 0): $\rho_m \approx a^{-3}$ Continuity Equation: $\rho \approx a^{-3}$
• Radiation dominated (w = 1/3): $\rho_r \approx a^{-4}$ Wavelength stretched with z
• Cosmological constant (w ≈ −1): $\rho_\Lambda = constant$ Constant Vacuum Energy
• Dark energy with (w < −1) e.g., w = −2: $\rho_{dm} \approx a^{+3}$ ....
– Energy density increases as is stretched out!
– Eventually would dominate over even the energies holding atoms together! ("Big Rip")

$$\rho_m(t) = \rho_{m,0}\, a^{-3}(t)$$
$$\rho_r(t) = \rho_{r,0}\, a^{-4}(t)$$
$$\rho_v(t) = \rho_v = \text{const.}$$

See Sections VII & XXVIII for Density Plot

In a mixed universe, different components ρ_m, ρ_r, ρ_Λ will dominate the global dynamics at different times →
Note in principle, it could be a function of time, density, etc

Density: Radiation, Matter, Λ



• Radiation density decreases the fastest with time
– Must increase fastest on going back in time
– Radiation must dominate early in the Universe
• Dark energy with w ≈ −1 dominates last; it is the dominant component now, and in the future

Models With Both Matter & Radiation ==>

However, to good approximation, assume that K = 0 and either **radiation** or **matter** dominate

"α" is the symbol for *proportional to*

		γ-dom	m-dom	Λ-dom
		$a(t)$	$\alpha\ t^{1/2}$	$\alpha\ t^{2/3}$
• Matter (m) dominated (w = 0):	$\rho_m\ \alpha\ a^{-3}$		$\alpha\ t^{-3/2}$	$\alpha\ t^{-2}$
• Radiation (γ) dominated (w = 1/3):	$\rho_\gamma\ \alpha\ a^{-4}$		$\alpha\ t^{-2}$	$\alpha\ t^{-8/3}$

• Cosmological Constant (Λ) (w = -1): ---> $\Lambda \approx a^{-\lambda t}$

Density of radiation today is mostly determined by the Temp of CMB.

$$z_{eq} \simeq 3612\, \Theta_{2.7}^{-4} \left(\frac{\Omega_{m0} h^2}{0.15}\right) \qquad \Theta 2.7 = T_{cmb}/2.75mK$$

$$T_{eq} = T_{CMB}(1 + z_{eq}) \qquad T_{eq} \simeq 5.65\, \Theta_{2.7}^{-3}\, \Omega_{m0} h^2\ \text{eV}$$

w = 1/3 radiation dominated $a(t) \propto t^{1/2}$
w = 0 matter dominated $a(t) \propto t^{2/3}$
w = -1 vacuum dominated $a(t) \propto e^{Ho\ t}$

<u>In 1922 Friedmann–Lemaître–Robertson–Walker (FLRW)</u> proposed a Relativistic Space-Time Metric that is the basis for an exact solution of Einstein's field equations of General Relativity; it is based on the assumption of a **homogeneous, isotropic, and expanding** (or otherwise, contracting) universe. The general form of the metric follows from the assumption of **homogeneity and isotropy** of space in the universe; Under these set of assumptions, Einstein's field equations are only needed to derive the scale factor of the universe as a function of time.

If we model the universe as a homogeneous, isotropic with spherical coordinates, we obtain the the Friedmann metric. **By defining a cosmic scale factor, "*a(t)*"**, which is a **function of time**. This scale factor parametrizes the **expansion of space**. The radius, r, is transformed to a comoving coordinate. **Furthermore, the radius of curvature** is also affected by cosmic expansion so it can be expressed in terms of the scale factor and a constant k

<u>The Friedmann–Lemaître–Robertson–Walker (FLRW) Relativistic Space-Time Metric in terms of "a" is:</u>

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dx^2}{1-\kappa\frac{x^2}{R^2}} + x^2 d\Omega^2\right] \qquad \text{where} \quad Kt = \frac{k}{a^2 t}$$

Note: "***a***" is NOT the acceleration, it is the Scale Factor *R(t)/R(t_0)*.

Based on this metric and its solution of the **<u>Einstein's Field Equations</u>** give the **<u>Two Friedmann Equations</u>**.

The assumption given the Field Equation: $R_{00} = T_{00}$

<u>The first Equation is:</u>

$$\frac{\dot{a}^2 + kc^2}{a^2} = \frac{8\pi G\rho + \Lambda c^2}{3} \qquad H^2 = \frac{8\pi\cdot G\cdot\rho + \Lambda\cdot c^2}{3} - \frac{k\cdot c^2}{a^2}$$

<u>The Second Equation is:</u>
the Evolution of the Cosmic Scale Factor, *a*.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}$$

where ***a* is the scale factor**, *G, Λ, and c* are universal constants. *G* is Newton's gravitational constant, *Λ* is the cosmological constant with dimension length^{-2}, and c is the speed of light in vacuum. *ρ* and *P* are the volumetric mass density and the pressure, respectively. *k* is constant throughout a particular solution, but may vary from one solution to another. The symbol "*a*" is defined as the scale factor which changes with time, *ρ and p* are the volumetric mass density and pressure. They may vary from one solution to another. The expansion of the universe (***ȧ/a***) can be measured**.**

In the Friedmann model, **H ≡ ȧ/a and is defined as the Hubble parameter, which evolves with time.**

<u>Hubble's Law, Expansion, and Redshift</u>

The Friedmann equation allows us to explain Hubble's discovery that recession velocity is proportional to the distance. The velocity of recession is given by $\vec{v} = d\vec{r}/dt$ and is in the same direction as $\vec{r}$, allowing us to write

$$\vec{v} = \frac{|\dot{\vec{r}}|}{|\vec{r}|}\vec{r} = \frac{\dot{a}}{a}\vec{r}.$$

The last step used $\vec{r} = a\vec{x}$, remembering that the comoving position $\vec{x}$ is a constant by definition. Consequently, Hubble's law $\vec{v} = H\vec{r}$ tells us that the proportionality constant, the Hubble parameter, should be identified as **H ≡ ȧ/a**

$$H = \frac{\dot{a}}{a}$$

and the value as measured today can be denoted with a subscript '0' as H_0. Because we measure Hubble's constant to be positive rather than negative, we know that the Universe is expanding rather than contracting.

We notice from this that the phrase Hubble's constant is a bit misleading. Although certainly it is constant in space due to the cosmological principle, there is no reason for it to be constant in time. In fact, using it as a more compact notation, we can write the Friedmann equation as an evolution equation for *H(t)*. as

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

It is best to use the phrase 'Hubble parameter' for this quantity as a function of time, reserving 'Hubble constant' for its present value. Normally the Hubble parameter decreases with time, for instance as the expansion is slowed by the gravitational attraction of the matter in the Universe.

Expansion and Redshift

The redshift of spectral lines that we **used to justify the assumption of an expanding Universe** can also be related to the scale factor. In this derivation we'll make the simplifying assumption that light is passed between two objects which are very close together, separated by a small distance *dr*. We've drawn the objects as galaxies, but we really mean two nearby points. According to Hubble's law, their relative velocity *dv* will be

$$dv = H\,dr = \frac{\dot{a}}{a}\,dr$$

where $d\lambda$ is going to be positive since the wavelength is increased. The time between emission and reception is given by the light travel time $dt = dr/c$, and putting all that together gives

$$\frac{d\lambda}{\lambda_e} = \frac{\dot{a}}{a}\frac{dr}{c} = \frac{\dot{a}}{a}dt = \frac{da}{a}$$

Integrate and we find that $\lambda = \ln a + constant$, that is $\lambda \propto a$

where λ is now the instantaneous wavelength measured at any given time.

Although as we've derived it this result only applies to objects which are very close to each other, it turns out that it is completely general. It tells us that as space expands, wavelengths become longer in direct proportion. One can think of the wavelength as being stretched by the expansion of the Universe, and its change therefore tells us how much the Universe has expanded since the light began its travels. For example, if the wavelength has **doubled**, the Universe must **have been half its present size** when the light was emitted. Redshift observed is the wavelength from the emitted source.

The redshift as defined in the equation below is related to the scale factor by

$$1 + z = \frac{\lambda_{obs}}{\lambda_{em}} = \frac{a(t_{obs})}{a(t_{em})} \qquad z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}$$

where λ_{em} and λ_{obs} are the wavelengths of light at the points of emission (the galaxy) and observation (us).

In order to solve the Friedmann Equation, **we need to define the behavior of the mass/energy density, ρ(a)** of any given mass/energy component. Recall the basic

General Relativity paradigm relating to Cosmology:

Density Determines the Expansion
Expansion Changes the Density

Λ Density Parameter

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{crit}}$$

Matter Density Parameter

$$\Omega_M = \frac{\rho_M}{\rho_{crit}}$$

Density Components: Each component will lead to a different evolution in redshift and a different Model

Matter, Radiation, Λ: $\rho_m(t) = \rho_{m0}\cdot a^{-3}(t)$ $\quad \rho_{rad}(t) = \rho_{rad0}\cdot a^{-4}(t)$ $\quad \rho_\Lambda(t) = \rho_\Lambda = constant$

$$\rho_0 = \frac{3\cdot H_0^2}{8\cdot\pi\cdot G}$$

Seconds in a Billion (Giga) Years, Gyr

$$Gyr := 3600\cdot 24\cdot 365.24\cdot 10^9\cdot s = 3.156\times 10^{16}\,s$$

MegaParSec (Mpc)

$$Mpc := 3\cdot 10^{19}\,km$$

Estimate of Hubble H_0

$$H_0 := 73\,\frac{km}{s}\cdot(Mpc)^{-1}$$

$$\Omega_M = \frac{8\pi G\cdot\rho}{3\cdot H_0^2} \qquad \Omega_\Lambda = \frac{\Lambda\cdot c^2}{3\cdot H_0^2}$$

When $a_0 = 1$

$$H_0^2\cdot\Omega_\Lambda = \Lambda\cdot c^2\,3$$

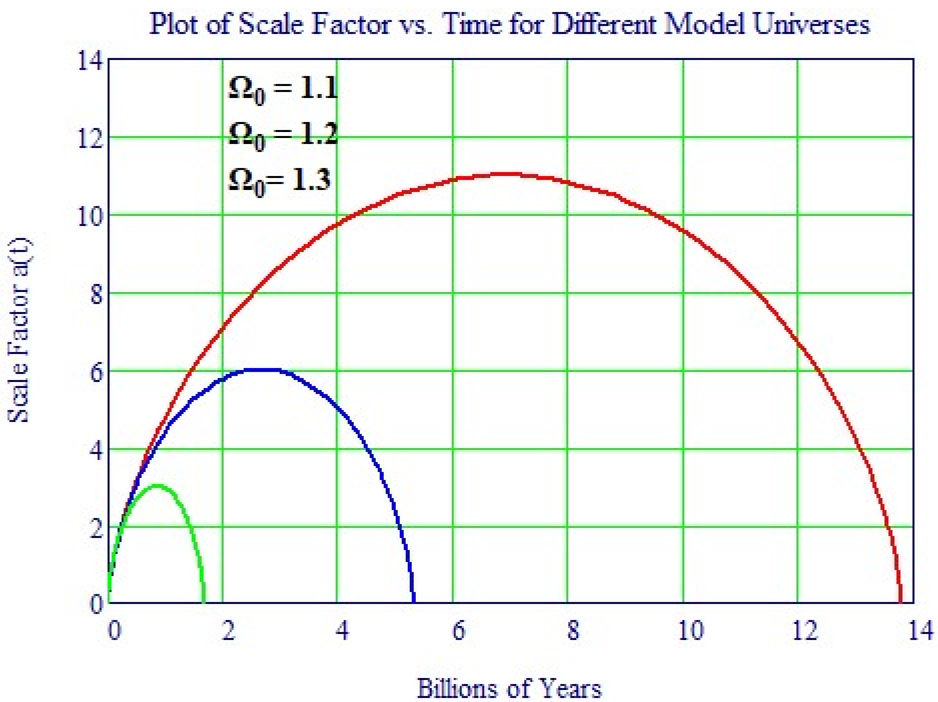
Models in Cosmology

In General: $\frac{8\pi G\rho}{3} = H_0^2\left(\Omega_{\Lambda,0} + \Omega_{m,0}a^{-3} + \Omega_{\gamma,0}a^{-4}\right)$

$\frac{1}{H_0} = 13.023 \cdot Gyr$

Example of Models

Einstein de Sitter Matter Only (γ, $\Lambda = 0$) Model **See Section VIII**.



Consider Several Simple Models

Refer to Section VIII for Model Details

- $k = 0$, matter dominated, **Einstein de Sitter**
- $k = 0$, radiation dominated
- $k < 0$, $\rho = 0$, Milne Model
- $k < 0$, $\rho > 0$
- $k > 0$
- Λ dominated

k is the curvature of space

w=1/2 radiation dominated $a(t) \propto t^{1/2}$

w=0 matter dominated $a(t) \propto t^{2/3}$

w = -1 vacuum dominated $a(t) \propto e^{H_o t}$

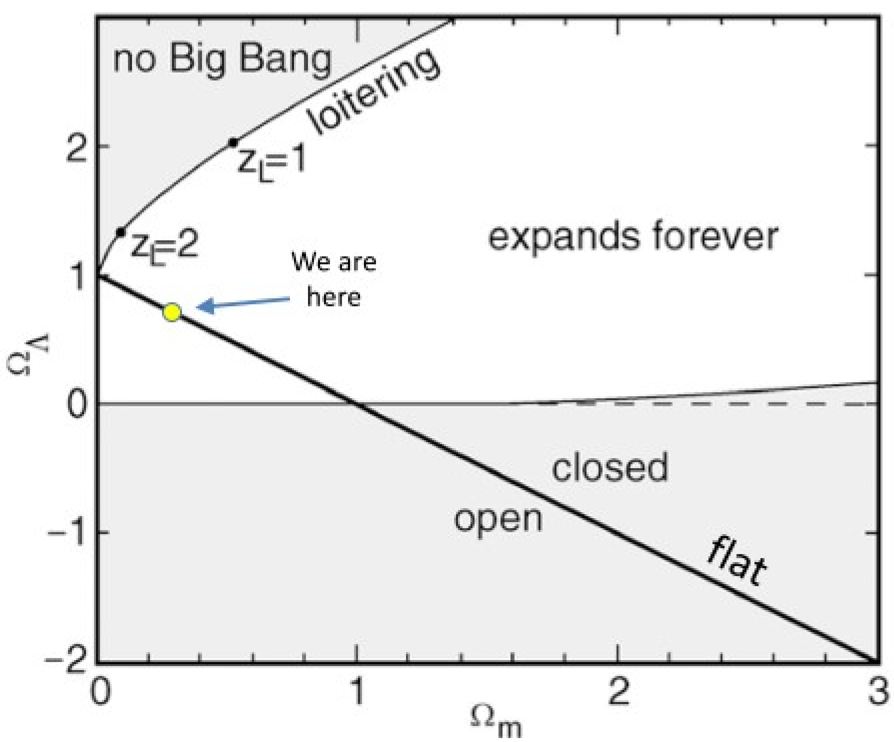
De Sitter Universe has a constant curvature surface embedded in Minkowski space-time (two-dimensional case)

The Milne Model Universe is simply a piece of Minkowski spacetime described in expanding coordinates.

Where dr^2 is transformed to $d\chi^2$

$$ds^2 = dt^2 - t^2(d\chi^2 + \sinh^2\chi\, d\Omega^2)$$

Classification of Models



(Ignoring Ω_{rad}, since it is negligible for most of the history of the universe)