

## V. Distances in Cosmology: The Basic Goal of Cosmology and Hubble's Law

**Hubble's law**, also known as the **Hubble–Lemaître law**, is the observation in physical cosmology that galaxies are moving away from Earth at speeds proportional to their distance. In other words, the farther they are, the faster they are moving away from Earth. The velocity of the galaxies has been determined by the change in the wavelength, redshift ( $z$ ), is a shift of the light they emit toward the red end of the visible spectrum.

The announcement of [Hubble's law in 1929](#) marks the [birth of Observational Cosmology](#). It is considered [the first observational basis for the expansion of the universe](#), and today it serves as one of the pieces of evidence most often cited in support of the  $\Lambda$ CDM model.

**Hubble's Law:** The motion of astronomical objects due solely to this expansion is known as the Hubble flow. It is described by the equation  $v = H_0 \cdot D$ , with  $H_0$  the constant of proportionality—the Hubble constant—between the **"proper distance"  $D$**  to a galaxy. The proper distance,  $D$ , can change over time, unlike the comoving distance, and its **speed of separation  $v$** , that is, the **derivative of proper distance with respect to the cosmological time coordinate**. The proper distance can also be defined as the separation between two objects at a **specific moment (simultaneously)** in cosmological time.

Suppose  $R(t)$  (or  $a(t)$ ) is called the **scale factor** of the universe, and increases as the universe expands in a manner that depends upon the cosmological model selected.  $t_0$  is some reference time,  $t$ . Its meaning is that all measured **proper distances  $D(t)$**  between co-moving points increase proportionally to  $R$  selected. All measured proper distances  $D(t)$  between co-moving points increase proportionally to  $R$ . (The co-moving points (gravitationally bound) are not moving relative to each other except as a result of the expansion of space.)

### Various Measures of Distance. Refer to Sections V and IX.

**Flux is the amount of energy from a source in  $W/m^2$ . Luminous flux,  $L_m$ , is a measure of the perceived power of visible light produced by a light source or light fitting. Its value is independent of an observer's distance from an object.** The luminous flux accounts for the sensitivity of the eye by weighting the power at each wavelength with the **luminosity function,  $lx$** , which represents the eye's response to different wavelengths.  $L_m = lx / Area$

<u>Scale Factor, <math>a(t)</math></u>	<u>Hubble Parameter, <math>H(t)</math></u>	<u>Proper Distance</u>	<u>Comoving Distance (<math>z</math>)</u>	<u>Luminosity Distance, <math>D_L</math></u>
$\frac{D(t)}{D(t_0)} = \frac{R(t)}{R(t_0)}$	$\frac{H}{H_0} = H(t) = \frac{dR(t)}{dt R(t)}$	$s(t) = a(t) \cdot r$	$D_c = D_H \int_0^z \frac{1}{E(z)} dz$	$d_L = \sqrt{\frac{L}{4\pi f}}$

#### Hubble Unit:

$$D_H = c \cdot H_0 \quad D_c = \frac{s(t)}{a} \quad E(z) = \sqrt{\Omega_k \cdot (1+z)^2 + \Omega_{0m} \cdot (1+z)^3 + \Omega_{0r} \cdot (1+z)^4 + \Omega_{0\Lambda}}$$

### The parameters that appear in Hubble's law, velocities and distances, are not directly measured.

In reality we determine, say, a **supernova brightness**, which provides information about its distance, and the redshift  $z = \Delta\lambda/\lambda$  of its spectrum of radiation. **Hubble correlated brightness** and parameter  $z$ .

### Calculating Luminosity Distance versus Redshift in FLRW Cosmology via Homotopy Perturbation Method

$$d_L = (L/4\pi f)^{1/2} \quad d_L = c(1+z) \int_0^z \frac{dz'}{H_0 E(z')}$$

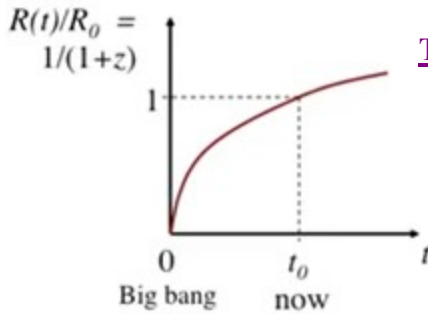
$$H^2 = H_0^2 \{ \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda \}$$

**So far, we've found out how to compute different cosmological models.**  
**But what good are they?**

**The basic goal of cosmology is to figure out in what model universe do we live.**

Models are basically distinguished by their history of the expansion rate, how their scale factor changes as a function of time.

If we can figure out which curve of those we live on, we know we'll know about cosmological parameters.



**This relation is arguably the single most important equation in Cosmology**

$$R_0 dr = \frac{c}{H(z)} dz \quad R(t)/R_0 = 1/(1+z)$$

$$= \frac{c}{H_0} [(1 - \Omega)(1 + z)^2 + \Omega_v + \Omega_m(1 + z)^3 + \Omega_r(1 + z)^4]^{-1/2} dz.$$

**Comoving Distance,  $D_C$**

$$D_{Cz}(z; \Omega_{0m}, \Omega_{0\Lambda}, \Omega_{0r}) := \int_{(1+z)^{-1}}^1 \frac{1}{\sqrt{\Omega_{0m} a\xi + \Omega_{0\Lambda} a\xi^4 + \Omega_{0r}}} da\xi$$

The expansion **Scale Factor  $R(t)$**  is simply related to **redshift,  $z$** , that is an observable quantity, and that's an easy part. The other axis is the time axis. Now unfortunately, this them galaxies do not carry gigantic clocks on them. Lookback time can only be inferred from a model. So it's very hard to figure out what is the look back time between us in some distant point, in a way that can be measured. So instead of that, what we do is we do **we transform coordinates,**

**instead of the look back time, we can use distance which is simply time multiplied by the speed of light.**

Distances in principle can be measured so we flipped the star Game and instead of expansion factor  $R(t)$ , we use the redshift, which is an observable quantity. And instead of the time we use a distance, which we can figure out how to measure in some way. NOTE: Different redshifts correspond not only to different times, but also to different places.

**So essentially, all cosmological tests boil down to this**

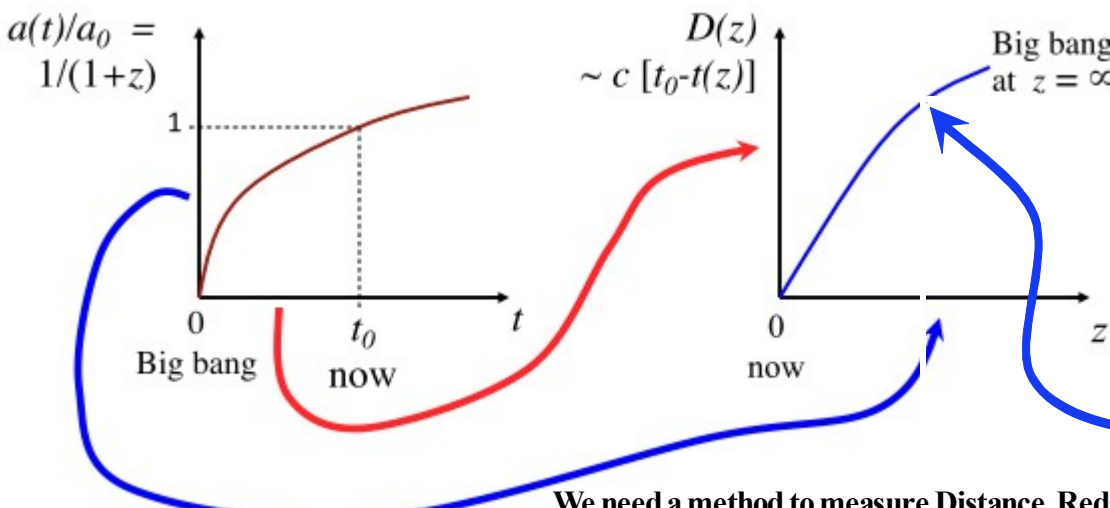
We have to **somehow measure a set of distances to a points as a function of redshift.** And because the whole thing just scales with Hubble constant, we only need to determine the shape of that curve.

So let's figure out how to measure distances in cosmology. A convenient unit of distance is Hubble distance, which is simply speed of the light divided by the Hubble constant. The Hubble constant has dimensions of one over time.

**The Basis of Cosmological Tests**

**From  $\Lambda$ CDM Scale Factor Past,  $a(t)$ , to Now( $t_0$ )**

**From Now  $D(z_0)$  at Distance  $D(z_{past})$**



**All Cosmological Tests** essentially consist of comparing some measure of (relative) distance,  $D(z) = c \cdot (t_0 - t_z)$  (or look-back time) to redshift,  $z$ . Absolute distance scaling is given by the  $H_0$ . So that all we need is the shape of the  $D(z)$  curve because it scales with  $H_0$ .

**We need a method to measure Distance. Redshift  $z$  can be measured. We can do this by measuring the Luminosity of an object. See XVII**

# Cosmological Tests: The Why and How

- Model equations are integrated, and compared with the observations
- The goal is to determine the global geometry and the dynamics of the universe, and its ultimate fate
- The basic method is to somehow map the history of the expansion, and compare it with model predictions
- A model (or a family of models) is assumed, e.g., the Friedmann-Lemaitre models, typically defined by a set of parameters, e.g.,  $H_0, \Omega_{0,m}, \Omega_{0,\lambda}, q_0, A$ , etc.
- Model equations are integrated, and compared with the observations

## V. Distances in Cosmology

A convenient unit is the Hubble distance or radius,  $D_H = c H_0 = 4.283 h_{70}^{-1} \text{ Gpc} = 1.322 \cdot 10^{28} h_{70}^{-1} \text{ cm}$   
 and the corresponding Hubble time,  $t_H = 1/H_0 = 13.98 h_{70}^{-1} \text{ Gyr} = 4.409 \cdot 10^{17} h_{70}^{-1} \text{ s} = 13.02 \text{ Gyr}$

At low z's, distance  $D \approx z D_H$ .

But more generally, the **comoving distance,  $D_C$**  to a redshift  $z$  is:

$$D_C = D_H \int_0^z \frac{1}{E(z)} dz$$

This integral is not solvable analytically and must be calculated numerically.

**The Hubble Parameter  $H/H_0 = E(Z)$ :** 
$$E(Z) = \sqrt{\Omega_k \cdot (1+z)^2 + \Omega_{0m} \cdot (1+z)^3 + \Omega_{0r} \cdot (1+z)^4 + \Omega_{0\Lambda}}$$

**The Hubble Parameter at a Given Distance is then:**

$$H(z) = H_0 E(z)$$

Note: All Distances and Time scale linearly with the Hubble Constant,  $H$

**The Curvature is determined by  $\Omega_k$ :** 
$$\Omega_k = 1 - \Omega_r - \Omega_m - \Omega_\Lambda$$

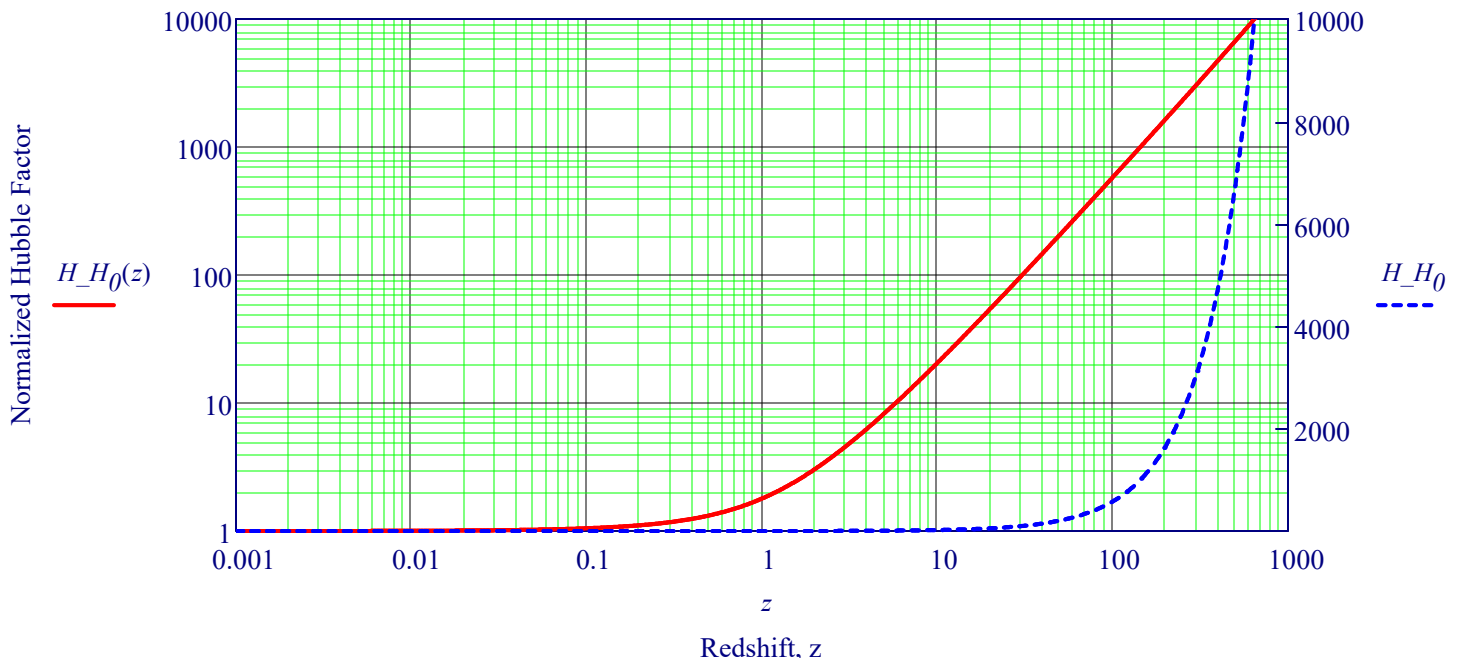
### $\Lambda$ -CDM Model Parameters (Flat Space $k=0$ )

$$\Omega_{r0} := 8.7 \cdot 10^{-5} \quad \Omega_{m0} := 0.317 \quad \Omega_{\Lambda0} := 0.683 \quad \Omega_{\Lambda0} + \Omega_{m0} + \Omega_{r0} = 1$$

### Dynamical Equation Specifying the Evolution of the Hubble Factor of Our Universe

$$\frac{H}{H_0} = \blacksquare H_{H_0}(z) := \sqrt{\Omega_{m0} \cdot (1+z)^3 + \Omega_{r0} \cdot (1+z)^4 + \Omega_{\Lambda0}}$$

Evolution of the Hubble Scale Factor vs. Redshift,  $z$  for Given  $\Omega_{m0}, \Omega_{\Lambda0}, \Omega_{r0}$



# Cosmological Distances: The Horizon Problem

## There are fundamentally Two Kinds of Coordinates in a GR cosmology:

### Proper coordinates: Stay Fixed, Space Expands Relative to Them.

Examples:

- Sizes of atoms, molecules, solid bodies
- Gravitationally bound systems, e.g., Solar system, stars, galaxies ...

### Comoving coordinates: Expand with the Universe.

Examples:

- **Unbound systems**, e.g., any two distant galaxies
- **Wavelengths of massless quanta**, e.g., photons
- **Stretches relative to the Proper Coordinates**

We introduce a **scale factor**, commonly denoted as  $R(t)$  or  $a(t)$ : a **spatial distance between** any two unaccelerated frames which move with their **comoving coordinates**.

**Computing  $a(t)$  and various derived quantities defines the cosmological models.**

This is accomplished by solving the Friedmann Equation

### 1. Proper Distances

We define a proper distance, as the distance between two events, A and B, in a reference frame for which they **occur simultaneously** ( $t_A = t_B$ ).

$$(ds)^2 = (cdt)^2 - a^2(t) \cdot \left[ \frac{dr^2}{(1 - kr^2)} + r^2 \cdot (d\theta^2 + \sin(\theta)^2 \cdot d\phi^2) \right]$$

and set  $d\theta = d\phi = 0$  and  $dt = 0$ , so that

$$s(t) = \int_0^s 1 ds' = a(t) \cdot \int_0^r \frac{1}{\sqrt{1 - kr^2}} dr$$

The proper distance has solution  $s(t)$ ,

where  $k$  is a curvature factor.

$$s(t) = a(t) \cdot \begin{cases} \frac{1}{\sqrt{k}} \sin^{-1}(r\sqrt{k}) & \text{for } k > 0 \\ r & \text{for } k = 0 \\ \frac{1}{\sqrt{|k|}} \sinh^{-1}(r\sqrt{|k|}) & \text{for } k < 0 \end{cases}$$

**In a flat universe, the proper distance to an object is just its coordinate distance,**

$$s(t) = a(t) \cdot r.$$

Because  $\sin^{-1}(x) > x$  and  $\sinh^{-1}(x) < x$ ,

### Universe Contracts (Closed) or Universe Expands (Open)

**in a closed universe ( $k > 0$ )**

**the proper distance to an object is greater than its coordinate distance,**

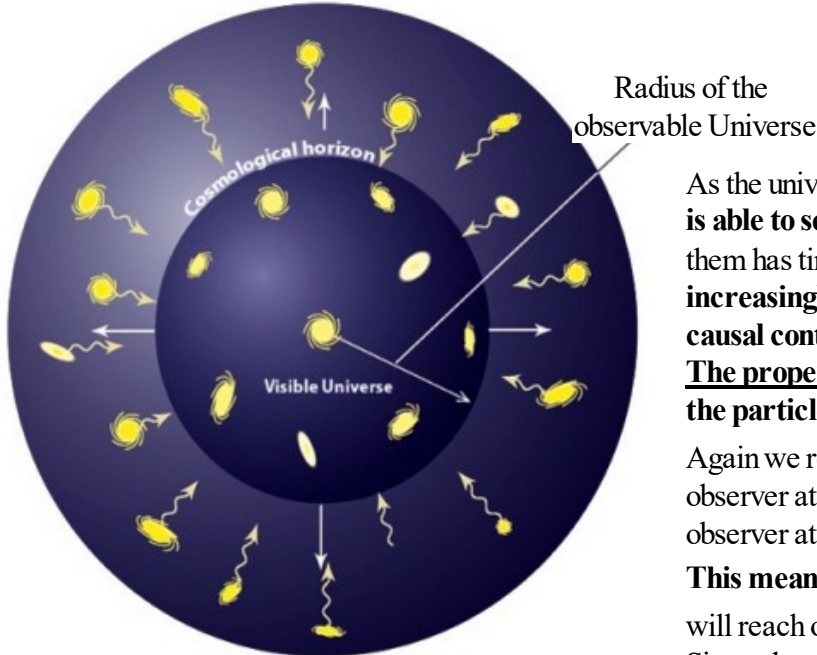
**while in an open universe ( $k < 0$ )**

**the proper distance to an object is less than its coordinate distance.**

The proper distance to the furthest observable point - the particle horizon - at time t is:

**Horizon Distance:**  $s_h(t)$

## The Horizon



As the universe expands and ages, **an observer at any point is able to see increasingly distant objects** as the light from them has time to arrive. This means that, **as time progresses, increasingly larger regions of the universe come into causal contact with the observer.**

**The proper distance to the furthest observable point, the particle horizon**— at time t is the **horizon distance,  $s_h(t)$ .**

Again we return to the Robertson-Walker metric, placing an observer at the origin ( $r = 0$ ) and let the particle horizon for this observer at time t be located at radial coordinate distance  $r_{hor}$ .

**This means that a photon emitted at  $t = 0$  at  $r_{hor}$  will reach our observer at the origin at time t.**

Since photons move along null geodesics,  $ds = 0$ . Considering only radially traveling photons ( $d\theta = d\phi = 0$ ), we find

$$\int_0^t \frac{1}{a(t)} dt = \frac{1}{c} \int_0^{\tau_{hor}} \frac{1}{\sqrt{1 + kr^2}} d\tau$$

$$\text{for } k=1 \left( c \cdot \int_0^t \frac{1}{a(t)} dt \right) \quad \text{for } k=0 \quad c \cdot \int_0^t \frac{1}{a(t)} dt$$

**The lookback time**  $t_L(z)$  to a source at any redshift z is the **time photons needed to travel with speed c** from the source to the observer at  $z = 0$ . In a homogeneous universe, this global quantity is just the sum of the small locally measured proper times  $dt$ . In terms of the scale factor  $a$  and  $H = d \ln(a)/dt$ , it is.

$$t_L = \int_0^1 \frac{1}{a' \cdot H(a')} da' \quad t_L(z) = \int_0^z \frac{1}{(1+z') \cdot H(z')} dz'$$

**If the scale factor evolves with time as  $a(t) = t^\alpha$ , we can see that the above time integral diverges as we approach  $t = 0$ , if  $\alpha > 1$ . This would imply that the whole universe is in causal contact.**

However,  $\alpha = 1/2$  and  $2/3$  in the radiation and matter-dominated regime, so there is a horizon.

**The proper distance from the origin to  $r_{hor}$  is given by:**

$$\text{for } k=0 \quad s_{hor}(t) = a(t) \cdot \int_0^{\tau_{hor}} \frac{1}{\sqrt{1 + kr^2}} d\tau = a(t) \cdot \int_0^t \frac{c}{a(t)} dt$$

So  $s_{hor}(t) = 2ct$  in the radiation-dominated era and  $s_{hor}(t) = 3ct$  in the matter-dominated era. Notice that these distances are **larger than ct, the distance travelled by a photon in time t**. How could this be? The reason lies in our definition of proper distance, as the distance between two events measured in a frame of reference where those two events happen at the same time.

To understand this, consider a photon emitted at comoving radial coordinate  $r_{hor}$  at time  $t = 0$ . We want to know what is the proper distance of that photon from **our** position, at  $r = 0$ , at a later time t. The coordinate of the photon at time t may be found by integrating

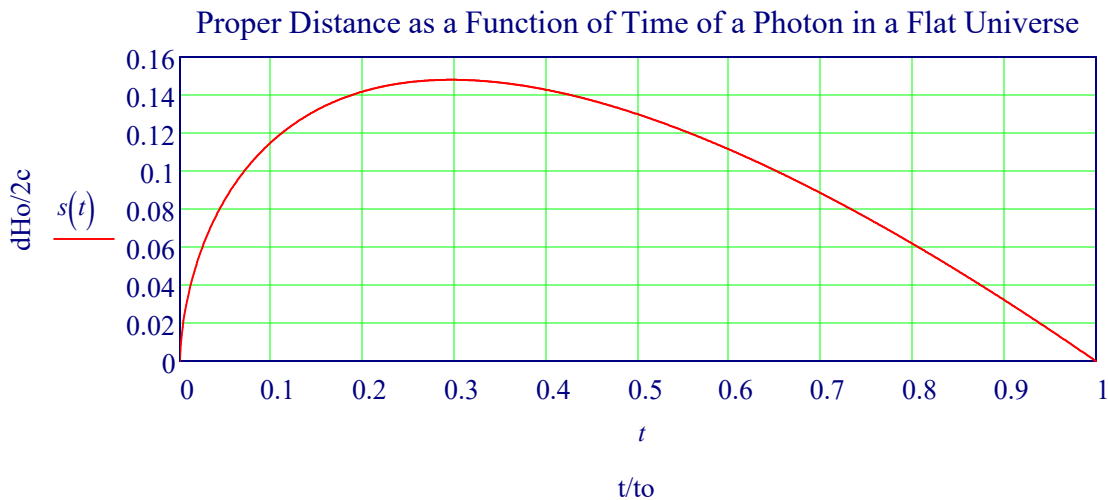
$$\int_0^t \frac{c}{a(t)} dt = \int_0^{\tau_{hor}} \frac{1}{\sqrt{1 + kr^2}} d\tau$$

As before, we consider zero curvature models. Substituting for  $a(t)$  we obtain:  $r = r_{hor} - \frac{2c}{H_0} \cdot \left(\frac{t}{t_0}\right)^{\frac{1}{3}}$

where  $t_0 = 2t_H/3$  is the present age of the universe. Recalling that  $r_{hor} = 2c/H_0$ , and that the proper distance in a flat universe is just  $s(t) = a(t) \cdot r$ , we find that the proper distance of the photon from Earth as a function of time is

$$s(t) := \frac{2c}{H_0} \cdot \left[ \left(\frac{t}{t_0}\right)^{\frac{2}{3}} - \frac{t}{t_0} \right] \quad \text{for } k=0$$

Proper distance as a function of time of a photon emitted from the present particle horizon at the time of the  $\Lambda$ CDM. The proper distance is expressed as a function of  $2c/H_0$ , the present horizon distance in a flat universe.



We can now see that **the initial expansion actually carried the photon away from Earth.** Although the photon's co-moving coordinate was always decreasing from an initial value  $r_{hor}$  towards Earth's position at  $r=0$ , the scale factor  $a(t)$ , (or  $R(t)$ ), increased so rapidly that **at first the proper distance between the photon and Earth increased with time.**

### Expansion and the Hubble's Law

Consider a point at a **comoving distance**  $x$ . At some time  $t$  it will be at a radial distance  $r(t) = a(t) x$ , where  $a(t)$  is the expansion factor. We will designate values for "here and now" with a subscript 0,  $t_0 = \text{now}$ , and  $a_0 = a(t_0) = 1$ . The recession velocity is:

$$v(r, t) = \frac{d}{dt} r(t) = \frac{da}{dt} x \equiv \dot{a} x = \frac{\dot{a}}{a} r \equiv H(t) r$$

Where  $H(t) := \frac{\dot{a}}{a}$  is the normalized expansion rate

$$\Delta v = v(r + \Delta r, t) - v(r, t) = H(t) \Delta r$$

Which is the same as the Hubble's law:  $v = H_0 D$

$H_0$  is the value of the **expansion rate here and now.**

Note that it is not a constant, but it depends on  $a(t)$ .

# Cosmological Distance Tests for Expanding vs. Static Universe

The James Webb Space Telescope (JWST) is capable of detecting objects at record-breaking redshifts,  $z \approx 15$ . This is a crucial advance for observational cosmology, as at these redshifts the

**differences between alternative cosmological models manifest themselves in the most obvious way.**

## Cosmological Tests

We shall focus primarily on the angular size–redshift relationship,  $\theta(z)$ , such as the Tolman surface-brightness test, the cosmological time dilation; number density–redshift relationship.

Cosmological models can be divided in two groups:

1. Expanding universes based on the Friedmann–Lemaître–Robertson–Walker (FLRW) metric with a time-dependent scale factor;
2. static universes based, e.g., on the metric including a scale factor in metric's time component.

The commonly accepted model of the first type is the standard  $\Lambda$ CDM cosmological model, which best fits observational data among other expanding-Universe models

## Compare Different Cosmological Models: Expanding Universe $\Lambda$ CDM vs. Static Comoving, $D_{CM}(z)$ , Luminosity Distance, $d_L(z)$ , Angular Diameter Distance, $D_A(z)$

$$\Omega_{0m} := 0.3 \quad \Omega_{0\Lambda} := 0.7 \quad D_{CM}(z) := \int_{(1+z)^{-1}}^1 \frac{1}{\sqrt{\Omega_{0m} \cdot a\xi + \Omega_{0\Lambda} \cdot a\xi^4}} da\xi$$

$$1 \text{ Gpc} = 3.26 \text{ GLightYear}$$

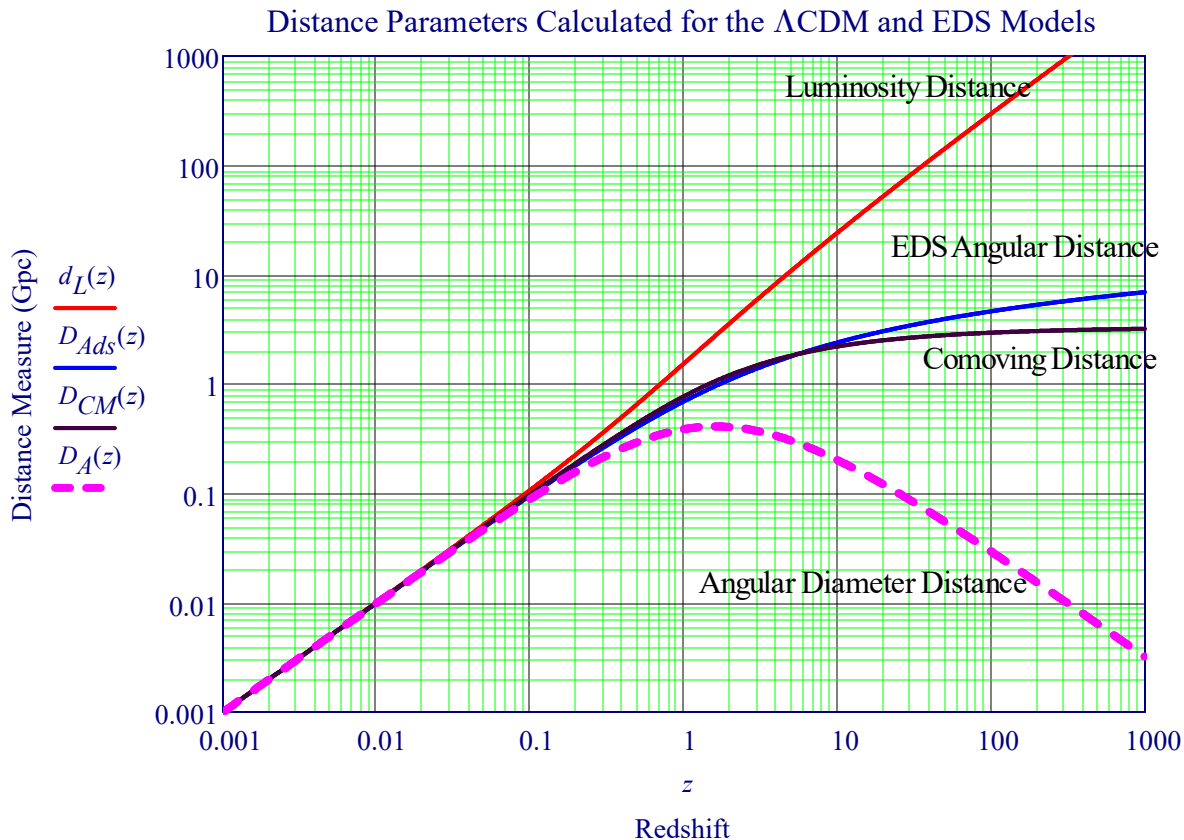
$$d_L(z) := (1+z) \cdot D_{CM}(z)$$

$$D_A(z) := \frac{D_{CM}(z)}{1+z}$$

### Flat Universe: Einstein de Sitter (EDS)

Angular Diameter Model

$$D_{Ads}(z) := \ln(1+z)$$



# Distance Formulas: Light Travel $D_{lt}$ , Present $D_{now}$ , Angular $D_A$ and Luminous $D_L$

Astronomy 140 Lecture Notes, Edward L. Wright, 2008, <https://astro.ucla.edu/~wright/intro.html>

$$D_{now} = R_o \psi = \int \frac{cdt}{a} = \int_{1/(1+z)}^1 \frac{cda}{a\dot{a}} \quad D_{lt} = \int cdt = \int_{1/(1+z)}^1 \frac{cda}{\dot{a}} \quad D_A = \frac{cZ(z) J([1 - \Omega_{tot}]Z^2)}{H_o (1+z)} \quad D_L = (1+z)^2 D_A$$

Fitting these formulae to the existing supernova data gives a set of contours of  $\Delta\chi^2$  as a function of  $\Omega_{m0}$  and  $\Omega_{\nu0}$  ( $\psi$  is the hyperbolic angle)

## Four Simple Cases and One Hard Case:

### For Each:

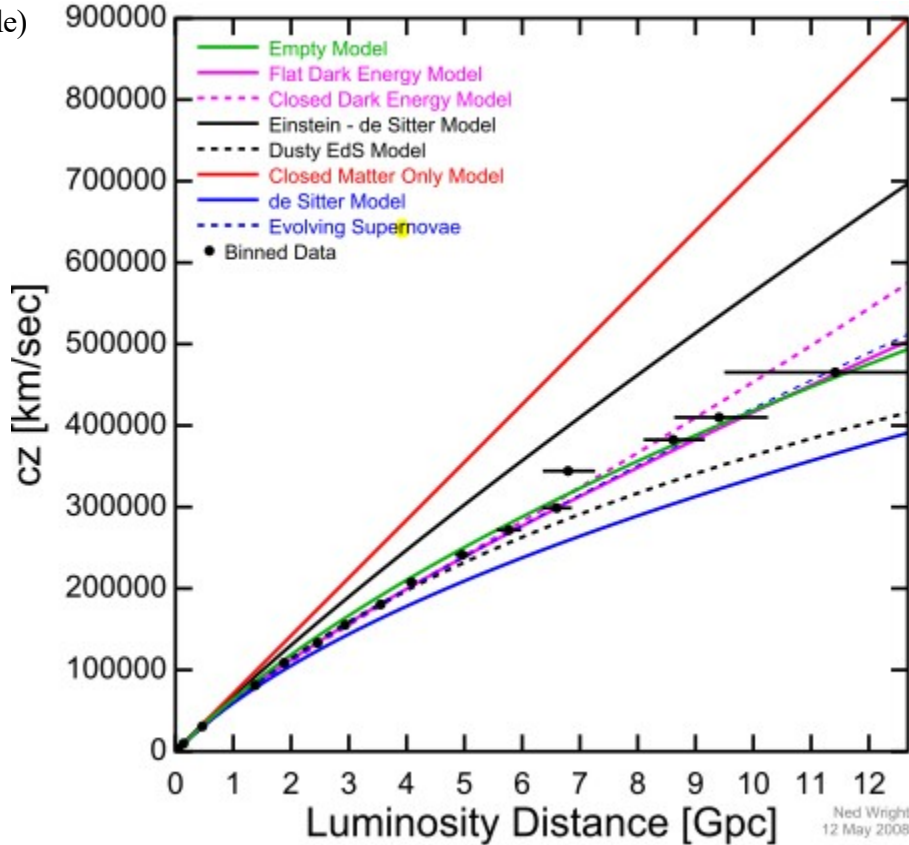
- $D_{now}$
- $D_{lt}$
- $D_A$
- $D_L$

### One of the 5 Cases

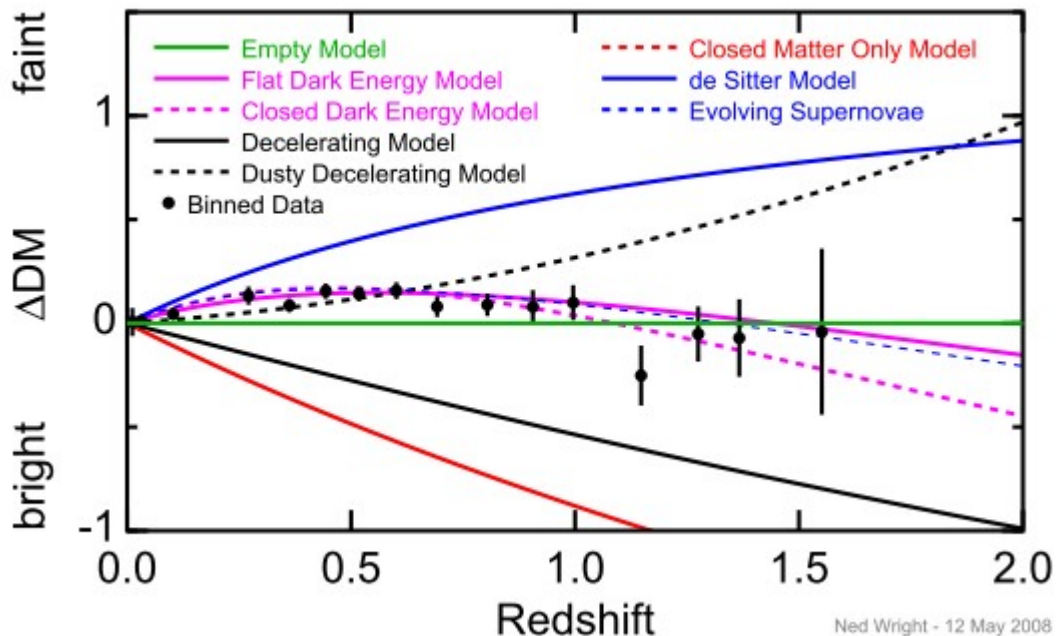
- $\Omega = 0$
- $\Omega_m = 1$
- $\Omega_r = 1$
- $\Omega_\nu = 1$

### Only $D_A$ and $D_L$

- $\Omega_m = 2$



Luminosity distance vs. redshift for high redshift Type Ia supernovae.



Distance modulus ( $m - M$ ) relative to an  $\Omega = 0$  model vs. redshift for high redshift Type Ia supernovae. The data points are binned values from the Kowalski et al. (2008, arXiv:0804.4142) union catalog of supernovae.