VI. Newtonian Energy Derivation of the Rate of Expansion, H

Consider a test particle of mass m as part of an expanding spherical shell of radius r & total mass M.

$$r(t) = a(t)\cdot x \qquad x = \frac{r(t)}{a(t)} \qquad v(r,t) = \frac{d}{dt}r(t) = \frac{da}{dt}x = \frac{da}{dt}\cdot\frac{r}{a} = \frac{a}{a}\cdot r = \frac{\dot{a}}{a}(t)\cdot r$$

Note: **"a"** is NOT the acceleration, it is the Scale Factor.

Note:
In the Newtonian Model Space is Euclidean and Gravity is a Force that causes massive bodies to accelerate, while in the Einsteinian View, Gravity is a manifestation of the Curvature of Spacetime. In the limit of weak spatial curvature or small $(v/c)^2$, the Newtonian View gives approximately the same results.

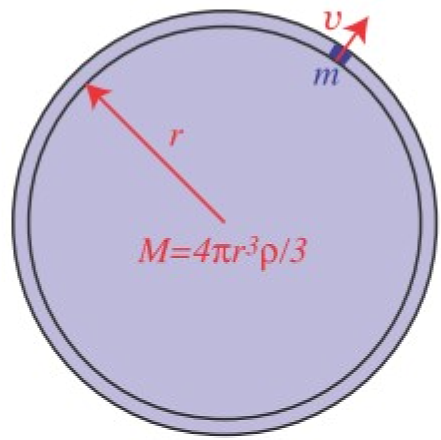
By Conservation of Energy, E = Constant

$$Energy = \frac{1}{2}m\cdot v^2 - \frac{GMm}{r}$$

$$\frac{1}{2}\left(\frac{d}{dt}r\right)^2 - \frac{GM}{r} - \frac{Energy}{m} = 0$$

$$M = \frac{4}{3}\pi r^3\cdot\rho \qquad r(t) = a(t)\cdot\frac{x}{r}$$

$$\frac{1}{2}\left(\frac{1}{r}\cdot\frac{d}{dt}r\right)^2 - \frac{G\cdot M}{r^3} - \frac{Constant}{r^2}$$



Rearrange to Friedmann Equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

Note: $\dot{a}$ is a contracting sphere if $\dot{a} < 0$
and k is proportional to Energy.

$$H^2 = \frac{8\pi\cdot G}{3}\cdot\rho - \frac{2\cdot Energy}{a^2}$$

- **If k = 0 (flat universe)**: $\dot{a}^2 > 0$, universe expands for ever, but as $a \to \infty$, $\dot{a} \to 0$
- **If k < 0 (open universe)**: $\dot{a}^2 > 0$, universe expands for ever, but $\dot{a}^2 \to c$
- **If k > 0 (closed universe)**: the expansion peaks when: $\dot{a}^2 = 0$.

The Two Friedmann Equations can be reduced to:

$$H^2 = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \qquad \rho_\Lambda \equiv \frac{\Lambda}{8\pi G_N} \qquad \rho_{tot} \equiv \rho + \rho_\Lambda$$

Expansion = Density - Curvature

ρ_Λ = Cosmological Constant Energy Density

ρ_{tot} = Total Energy Density

For a given value of H, there is a special value of the density which would be required in order to make the geometry of the Universe flat, that is, k = 0 . This is known as the critical density ρ_c

Note that to get the results in the FWLR form, we replaced the Energy Density term $\varepsilon_c(t)$ with the mass density, ρ.

$$\varepsilon_c(t) = \rho_c\cdot c^2 \qquad \rho_c \equiv \frac{3H^2}{8\pi G_N}$$

Sources of Matter and Energy

In General Relativity, all of the sources of matter and energy are included and contribute to the total energy density, ρ_{tot}. **The energy density today of each component is Normalized to the Critical Density, ρ_c,** (See below: $\Omega_{component}$) that is used in the definition of the corresponding "Omega parameter", **Ω.**

$$\Omega_{component} = \frac{\rho_{component}}{\rho_{c\_z0}}$$

Thus we have: $\mathbf{\Omega} = \Omega_{baryon} + \Omega_{cdm} + \Omega_{radiation} + \Omega_{DE}$

Here Ω_{baryon} is the baryon content, Ω_{cdm} is the amount of cold dark matter, $\Omega_{radiation}$ is the radiation content, and Ω_{DE} is the contribution from dark energy. If Ω = 1 that means the density is equal to the critical density, ρ_c, at z = 0, so we have a flat Universe (k = 0).